ATHEMATICS
R 25 1940

Mathematical Reviews

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January, 1940 TABLE OF CONTENTS Vol. 1, No. 1

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MATHEMATICAL REVIEWS

Published by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Those organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Application for entry as second class matter is pending at Lancaster, Pennsylvania

Mathematical Reviews

Vol. 1, No. 1 JANUARY, 1940 Pages 1-32

ALGEBRA

Venkatarayudu, T. The *7-15* problem. Proc. Indian Acad. Sci., Sect. A. **9**, 531 (1939). [MF 3]

The problem of placing $2n+1$ people around a table n times so that no two persons sit beside each other more than once is solved when $n=7$. *P. Scherk* (New York, N. Y.).

* **Dickson, Leonard Eugene. New First Course in the Theory of Equations.** John Wiley and Sons, Inc., New York, 1939. ix+185 pp. $1.75.

This book retains the flavor and spirit of the author's "First Course in the Theory of Equations," but it is written in a more expansive style with many illustrations either following or preceding the introduction of new ideas or topics, and with minute attention to details.

Extract from the preface.

Sispanov, Sergio. Generalización del teorema de Laguerre. Bol. Mat. **12**, 113–117 (1939). (Spanish) [MF 80]

Corliss, J. J. Upper limits to the real roots of a real algebraic equation. Amer. Math. Monthly **46**, 334–338 (1939). [MF 157]

The bounds provided here are sharper but more complicated than those given by Lagrange and Jean J. Bret. *O. Szász* (Cincinnati, Ohio).

Eagle, Albert. Series for all the roots of a trinomial equation. Amer. Math. Monthly **46**, 422–425 (1939). [MF 204]

The author gives power-series solutions for all roots of $z^{m+n}-pz^n+q=0$, m, n positive integers, p, q complex. References are given to McClintock [1895] and Birkeland [1920/21], but the important contributions of Nekrassoff [for example, Math. Ann. **29**, 1887] are not mentioned. Compare also A. J. Lewis [Nat. Math. Mag. **10** (1935) (abstr. Ph.D. thesis, University of Colorado, 1932)]. The standard tool for the power-series solution of the trinomial equation, the Lagrange expansion of x in $x=h+k\phi(x)$, is used by the present author. A difficulty exists in obtaining convergent series for all roots of the equation. The various methods of solving the equations by Lagrange expansions differ in the manner in which this difficulty is overcome. The author gives what seems to be a compact and usable method. *A. J. Kempner* (Boulder, Colo.).

Eagle, Albert. Series for all the roots of the equation $(z-a)^m=k(z-b)^n$. Amer. Math. Monthly **46**, 425–428 (1939). [MF 205]

As the author indicates, this equation can be transformed into a trinomial equation (let $z-a=x^n$). But it can be treated directly by reducing to a standard form $z^m=k(z-1)^n$, and applying Lagrange's expansion. In order to guarantee convergence of the series it is necessary to consider various cases according to the relations existing between m, n, k. This runs parallel to the treatment of trinomial equations [see preceding paper by the same author].

As an application, the solution of the general cubic by this method is given. *A. J. Kempner* (Boulder, Colo.).

Erdös, P. and Grünwald, T. On polynomials with only real roots. Ann. of Math. **40**, 537–548 (1939). [MF 93]

Es sei $f(x)$ ein Polynom mit nur reellen Wurzeln,

$$f(-1)=f(1)=0, \quad 0<f(x)\leqq f(\mu) \quad \text{für } -1<x<1,$$

wobei $-1<\mu<1$, so dass μ die Stelle des Maximums von $f(x)$ im Intervall $(-1, 1)$ bedeutet. Dann ist

$$\tfrac{2}{3}\frac{2f'(1)f'(-1)}{f'(1)-f'(-1)}\leqq\int_{-1}^{1}f(x)dx\leqq\tfrac{2}{3}\cdot 2f(\mu),$$

d.h. die "Fläche der Kurve" ist enthalten zwischen 2/3 der Fläche des "Tangentialdreiecks" und 2/3 der Fläche des "Tangentialrechtecks." Das Gleichheitszeichen wird dann und nur dann erreicht, wenn $f(x)$ vom 2-ten Grad ist, d.h. nur im wohlbekannten (Archimedischen) Fall des Parabelsegments. Die Verfasser geben einen Induktionsbeweis, der nicht ganz einfach ist. Durch Weglassen des Integrals ergibt sich, dass die Höhe des Tangentialdreiecks $\leqq 2f(\mu)$ ist; die Verfasser gehen von diesem, von G. Szekeres gefundenen Satz und dessen einfachem Beweis aus. Man kann bemerken, dass die erste Hälfte der Doppelungleichung sich auch mit der Methode von Szekeres beweisen lässt. Es hat nämlich $f(x)$, abgesehen von einem positiven konstanten Faktor, die Gestalt

$$f(x)=(1-x^2)\prod_{\mu}(\alpha_\mu-x)\prod_{\nu}(\beta_\nu+x),$$

wobei $\alpha_\mu\geqq 1$, $\beta_\nu\geqq 1$. Man erhält, mit Benutzung der Ungleichung zwischen dem arithmetischen, dem geometrischen und dem harmonischen Mittel von zwei Zahlen, dass

$$\begin{aligned}\int_{-1}^{1}f(x)dx&=\int_{-1}^{1}\tfrac{1}{2}(1-x^2)[\prod(\alpha_\mu-x)\prod_{\nu}(\beta_\nu+x)\\&\qquad+\prod_{\mu}(\alpha_\mu+x)\prod(\beta_\nu-x)]dx\\&\geqq\int_{-1}^{1}(1-x^2)[\prod(\alpha_\mu^2-x^2)\prod(\beta_\nu^2-x^2)]^{\frac{1}{2}}dx\\&\geqq\tfrac{4}{3}[\prod(\alpha_\mu^2-1)\prod(\beta_\nu^2-1)]^{\frac{1}{2}}=\tfrac{2}{3}[-f'(1)f'(-1)]^{\frac{1}{2}}\\&\geqq\tfrac{2}{3}\frac{2f'(1)f'(-1)}{f'(1)-f'(-1)}.\end{aligned}$$

G. Pólya (Zürich).

Collar, A. R. On the reciprocation of certain matrices. Proc. Roy. Soc. Edinburgh **59**, 195–206 (1939). [MF 161]

The paper is concerned with the rapid computation of the reciprocals (inverses) of matrices of special type. Let $M=(m_{ij})$ be a moment matrix

$$m_{i+j-1}=\int_a^b w(x)x^{i+j-2}dx,$$

where $w(x)$ is a weight function. Let

$$p_i(x)=p_{i1}+p_{i2}x+\cdots+p_{ii}x^{i-1}$$

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satisfy the orthogonal relations

$$\int_a^b w(x)p_i(x)p_j(x)dx=d_{ii}\delta_{ij}.$$

If P denotes the matrix (p_{ij}), then $PMP'=D$ is diagonal, and M^{-1} is computed from $M^{-1}=P'D^{-1}P$. Reciprocals of alternant (Vandermonde) matrices are also computed. Numerous examples are given. *C. C. MacDuffee.*

Abstract Algebra

Birkhoff, Garrett and Ward, Morgan. A characterization of Boolean algebras. Ann. of Math. **40**, 609–610 (1939). [MF 100]

The authors prove the following: Let L be a complete lattice (a lattice in which there exist unrestricted meets and joins), which is atomistic in the sense that, if a is an element of L for which $0<a<1$, there exist p, q in L such that $0<p\leqq a\leqq q<1$, while $0<x<p$ and $q<y<1$ have no solutions. Then L is a Boolean algebra if and only if each of its elements has one and only one complement.

H. Wallman (Princeton, N. J.).

Ward, Morgan. Ring homomorphisms which are also lattice homomorphisms. Amer. J. Math. **61**, 783–787 (1939). [MF 39]

If R is a commutative principal ideal ring with a unit, it determines a lattice in which the inclusion $a\supset b$ holds in L when a divides b in the ring R, while a and b are identified in L if they are associates in R. Problem: If R is homomorphic to R', what relation subsists between the corresponding lattices L and L'? Since every ideal is a principal ideal (m), R' is one of the residue class rings $R/(m)$. It is shown that the correspondence $a\rightarrow(a, m)$ is a lattice homomorphism of L to L', and that L' is isomorphic to the sublattice of L consisting of divisors of m. The proof uses the fact that L is isomorphic to the lattice of ideals of R, hence is distributive by the Ward-Dilworth theory of residuated lattices [Duke Math. J. **3**, 627–636 (1937); Proc. Nat. Acad. Sci. U. S. A. **24**, 162–164 (1938)]. *S. MacLane.*

Ward, Morgan and Dilworth, R. P. The lattice theory of ova. Ann. of Math. **40**, 600–608 (1939). [MF 99]

An "ovum" (E. T. Bell, A. H. Clifford) is a commutative, associative system with unit; an "ideal" of an ovum is a subset containing all multiples of its elements. The ideals of any ovum form a "residuated lattice" in the sense of the authors [Residuated lattices, Trans. Amer. Math. Soc. **45**, 335–354 (1939)]. The authors prove various abstract versions of theorems on polynomial arithmetic, such as: If the lattice of ovoid ideals is modular and if the ascending chain condition holds, then all the decomposition theorems of E. Noether for the ideals of commutative rings hold.

G. Birkhoff (Cambridge, Mass.).

Hopkins, Charles. Rings with minimal condition for left ideals. Ann. of Math. **40**, 712–730 (1939). [MF 106]

An *MLI* ring is a ring A which satisfies the descending chain condition for left ideals. These rings are discussed with respect to (a) their radical R, their idempotent (in particular, principal idempotent) elements, and the Peirce decompositions belonging thereto; (b) two ideals M, N which are determined by the ring as the sum of all minimal left ideals and the sum of all nilpotent such left ideals.

The radical is an ideal; it is nilpotent, and so is every subring of A which consists of nilpotent elements. Of the numerous results pertaining to Peirce decompositions we mention the following two: (a) if the ring A has a right unit or a left unit, then the ascending chain condition for left ideals follows; (b) if $R^2=0$, then the subring annihilated on both sides by a principal idempotent e is the same for all e.

The ideal N satisfies $N^2=0$. There exist direct complements L of N in M; a left ideal of L is also a left ideal in A, and L is semisimple. If e is the unit-element of L, then we have $L=eAe$. Every such complement is obtained by passing from e to $e+d$, where d is in the product NM.

Any *MLI* ring is the sum (not necessarily direct) of an idempotent ring $A^k=A^{k+1}$ and a nilpotent ring which may be chosen as the subring annihilated on both sides by a principal idempotent of A.

For nilpotent rings the descending chain condition for right ideals and even for subrings is derived; the *MLI* condition is here so strong that every element is of finite (and without much loss of generality, even of prime power) order in the additive group.

Some of the results, in particular the one about the ascending chain condition, remain valid for rings with operators. *M. Zorn* (Los Angeles, Calif.).

Everett, C. J., Jr. Annihilator ideals and representation iteration for abstract rings. Duke Math. J. **5**, 623–627 (1939). [MF 177]

This paper studies iterated left and right representations of an abstract ring Γ. Any fixed element a in Γ determines by left multiplication an operator $\alpha(x)=a\cdot x$ on the elements x of Γ. The set L of operators α constitutes the left representation of Γ, and is isomorphic to the residue class ring of Γ modulo a left annihilator ideal. The ring Γ is called left definite if the homomorphism $a\rightarrow\alpha$ of Γ to L is an isomorphism. It is proved that the ring resulting from the iteration of l left and r right representations in any mixed order is independent (up to isomorphism) of the order of the iteration. Rings with units and the representation of the center are considered. If Γ is both left and right definite, $ax=xb$ for every x implies $a=b$. Furthermore the endomorphism ring of an abelian group is proved commutative [see Everett, Bull. Amer. Math. Soc. **45**, 274–279 (1939)]. The proofs are highly formalized. Some could be simplified by exploiting known properties of ideals in residue class rings. Troublesome omissions: to Theorem 5, add "with $l+k=n$"; to Theorem 6, add "for some n."

S. MacLane (Cambridge, Mass.).

Hebroni, P. Sur les inverses des éléments dérivables dans un anneau abstrait. C. R. Acad. Sci. Paris **209**, 285–287 (1939). [MF 231]

This note continues the author's previous studies in an abstract complete normed ring R [Compositio Math. **5**, 403–429 (1937); **6**, 258–284 (1938)]. There is postulated a transformation a', called "differentiation," on a subring R_d of R to R with the properties $(a\pm b)'=a'\pm b'$, $(ab)'=a'b+ab'$. An element $a\varepsilon R$ is called perfect if the inverse a^{-1} exists and if a, $a^{-1}\varepsilon R_d$. The following is a typical result: If a^{-1} exists and $a\varepsilon R_d$ and if the initial value c (constant of "integration") of a is perfect, then a is also perfect.

A. D. Michal (Pasadena, Calif.).

Perlis, Sam. Maximal orders in rational cyclic algebras of composite degree. Trans. Amer. Math. Soc. **46**, 82–96 (1939). [MF 13]

This paper deals with the explicit construction of maximal orders of normal primary division algebras D over the

rational number field R. The author supposes that the degree n of D is either π^e (π an odd prime) or $2^e>2$, where D shall not be ramified at 2 and q_∞. Then D has class number 1. To exhibit maximal orders special splitting fields Z are constructed. Suppose that D is ramified at $q_1, \cdots, q_s$ and has the q_i-indices n_i. Then there exist infinitely many cyclic fields Z of degree n over R such that (a) $D=(Z, S, \sigma)$, $\sigma=\Pi_{i=1}^{s} q_i^{n/n_i}$; (b) Z has as conductor a prime $\mathfrak{p}$ such that $\mathfrak{p}\equiv 1 \pmod{n}$, $(\mathfrak{p}, \sigma)=1$; (c) $q_1, \cdots, q_s$ are completely inert in Z; (d) $\sigma\equiv r^n \pmod{\mathfrak{p}}$. This existence proof requires the construction of suitable congruence groups in R and makes use of the theorem on arithmetical progressions. Since every Z is contained in a suitable cyclotomic field, it follows that

$$\mathfrak{p}=(\beta)^n, \quad \beta=\prod_{i=0}^{h-1}(1-\xi^{g^i}), \quad N_{Z|R}(\beta)=p,$$

where $h=(p-1)n^{-1}$ and g is a primitive root of p, $\xi^p=1$. Now let Z_0 denote the maximal order of some fixed Z. Then

$$I=Z_0+uZ_0+\cdots+u^{n-1}Z_0$$

is an order of D, where u generates the automorphism S of $Z|R$ within D, $u^n=\sigma$. The author constructs exactly n maximal orders $I(\lambda)$ in terms of module representations. The maximal orders $I(\lambda)$ depend on a parameter which is taken to be one of the n solutions of the congruences $\lambda^n\equiv\sigma \pmod{\mathfrak{p}}$, $\lambda\equiv 0 \pmod{\sigma}$. Setting

$$y=(\lambda-u)\beta^{-1}, \quad \tau_j=\prod_{i=1}^{s} q_i^{e_i}, \quad e_i=[j/n_i], \quad j=1, \cdots, n-1,$$

one finds

$$I(\lambda)=Z_0+y\tau_1^{-1}Z_0+\cdots+y^{n-1}\tau_{n-1}^{-1}Z_0.$$

In order to establish that the rings $I(\lambda)$ are maximal orders containing I, it is only necessary to compute their discriminants. The choice of Z and the character of decomposition of the q_i immediately yield the assertion. Finally the author examines the discriminants of direct products of normal simple algebras. *O. F. G. Schilling* (Chicago, Ill.).

Nakayama, Tadasi. On Frobeniusean algebras. I. Ann. of Math. **40**, 611–633 (1939). [MF 101]

An algebra A possessing a unit element is a Frobeniusean algebra if the first regular representation R and the second regular representation S of A are equivalent. If R can be transformed into S by means of a symmetric non-singular matrix, then A is symmetric [cf. R. Brauer and C. Nesbitt, Proc. Nat. Acad. Sci. U. S. A. **23**, 236–240 (1937); C. Nesbitt, Ann. of Math. **39**, 634–658 (1938)]. Furthermore, the author calls an algebra quasi-Frobeniusean if R and S have the same distinct indecomposable constituents (perhaps with different multiplicities). The aim of the paper is the study of the properties of these and some related types of algebras. If M is any set of elements of an algebra A, then the left annihilators of M form a left ideal $l(M)$, the right annihilators a right ideal $r(M)$. The main result is the following: The algebra A is quasi-Frobeniusean if and only if (α) $l(r(\mathfrak{l}))=\mathfrak{l}$, $r(l(\mathfrak{r}))=\mathfrak{r}$ is satisfied for every left ideal $\mathfrak{l}$ and every right ideal $\mathfrak{r}$; the algebra A is Frobeniusean if and only if besides (α) the condition (β) $(\mathfrak{l}:F)+(r(\mathfrak{l}):F)=(A:F)$, $(\mathfrak{r}:F)+(l(\mathfrak{r}):F)=(A:F)$ is satisfied for every $\mathfrak{l}$ and every $\mathfrak{r}$. If these conditions (α) (or (α) and (β)) are satisfied for every nilpotent simple $\mathfrak{l}$ and $\mathfrak{r}$ as well as for the radical N and the 0-ideal (0), then they hold for every $\mathfrak{l}$ and every $\mathfrak{r}$. It also is sufficient to assume (α) and (β) for the cases that $\mathfrak{r}=\mathfrak{l}$ is either a nilpotent simple two-sided ideal, or $\mathfrak{r}=\mathfrak{l}=N$, or $\mathfrak{r}=\mathfrak{l}=(0)$, in order to assure that A is Frobeniusean. In the case of quasi-Frobeniusean algebras, the corresponding extension of the theorem can only be stated for algebras over an algebraically closed ground field F. In connection with these results it is shown that $r(N^\nu)=l(N^\nu)$, $\nu=1, 2, \cdots$, if A is quasi-Frobeniusean. Any two-sided ideal $\mathfrak{z}$ of a Frobeniusean algebra A which is a principal left ideal $\mathfrak{z}=Ac$ satisfies $\mathfrak{z}=cA$, so that it is also a principal right ideal. The residue class algebra $A/\mathfrak{z}$ of a Frobeniusean algebra is Frobeniusean again if and only if $r(\mathfrak{z})$ is such a principal ideal. If A is symmetric, then $r(\mathfrak{z})=l(\mathfrak{z})$ for every two-sided ideal $\mathfrak{z}$. Here $A/\mathfrak{z}$ is symmetric again if and only if $r(\mathfrak{z})$ is of the form Ac, where c lies in the center of A. If $\mathfrak{l}_0$ is a left sub-ideal of a left ideal $\mathfrak{l}$ of a symmetric algebra, then the representation of A defined by the left module $\mathfrak{l}/\mathfrak{l}_0$ is equivalent to the representation defined by the right module $r(\mathfrak{l}_0)/r(\mathfrak{l})$.

R. Brauer (Toronto, Ont.).

MacLane, Saunders. Steinitz field towers for modular fields. Trans. Amer. Math. Soc. **46**, 23–45 (1939). [MF 10]

Let K be a field of characteristic p. The field K is said to have a separating transcendence base (s.t.b.) T over a subfield L if and only if T is a transcendence base over L and such that K is separable over $L(T)$. Fields K having s.t.b. over L can be characterized as follows: A necessary and sufficient condition is that the elements of K can be well ordered in such a way that every $b\in K$ is either transcendental or separable algebraic over the field K_b obtained by adjoining to all elements prior to b in the well ordering. In order to describe perfect fields over their respective prime fields P, it is necessary to introduce the notions of p-independent sets X and p-bases of K. A set $X\subset K$ is called p-independent, if $K^p(X')\subset K^p(X)$ whenever $X'\subset X$. A set $X\subset K$ is a p-basis of K if X is p-independent in K and $K=K^p(X)$. Now let S_n be the field of all $\alpha\in K$ with α^{p^n} separable over $L(T)$, then

$$S_0\subset S_1\subset S_2\subset\cdots, \quad K=\sum S_i$$

(Steinitz tower). If K is a perfect field, then one can prove: If K is a perfect field containing a perfect field $L=P$ relatively algebraically closed in K and if K has a transcendence base T over L, then (i) the nth field S_n of the Steinitz tower has the s.t.b. $T^{p^{-n}}$ over P; (ii) $S_n=P(S_{n+1}^p)$. To obtain information on non-perfect fields, the author supposes the following hypothesis to hold: P is a perfect subfield of K; X is a p-basis of K; L is the field of elements of K algebraic over $P(X)$. Now let K have a finite degree of transcendentality over L. Then each field S_n of the Steinitz tower has a s.t.b. T_n over L and hence also a s.t.b. $X+T_n$ over P; each basis T_n has the same number of elements as does T. Property (ii) need not hold for every Steinitz tower. However, under the hypothesis and if the transcendence base T for K over L is finite, then there exists a set of subfields M_k of K,

$$M_0\subset M_1\subset M_2\subset\cdots, \quad K=\sum M_k,$$

such that (i) each M_k has a s.t.b. T_k' over L, (ii) $M_k\subset L(D_{k+1}^p)$, where $D_{k+1}=M_{k+1}-M_k$ is the set of elements in M_{k+1} but not in M_k. In particular, if T consists of one element then $S_k=L(S_{k+1}^p)$. It is shown by an example that the latter statement need not be true if T contains more than one element. A rather intricate example shows that the fields S_k of a Steinitz tower need not all have s.t.b. over L if the field K has infinite degree of transcendentality over L. *O. F. G. Schilling* (Chicago, Ill.).

Arf, Cahit. Untersuchungen über reinverzweigte Erweiterungen diskret bewerteter perfekter Körper. J. Reine Angew. Math. **181**, 1–44 (1939). [MF 5]

The goal of this paper is to give an arithmetic proof of Artin's conductor formula for normal algebraic number fields. The author remarks that it suffices to prove the formula for local abelian extensions. The first three sections deal with an analysis of the generating equations of totally ramified extensions $K|k$. The author generalizes the theory of Marc Krasner [Sur la primitivité des corps p-adiques, Mathematica, Cluj **13**, 72–191 (1937)]. The assumption that the residue field Ω of k is an arbitrary field of characteristic p necessitates the consideration of a set of multiplicative representatives of Ω in k. The exponent of the conductor

$$E=e^{-1}[e\chi(1)-\chi(V_0)+p^{R_1}\chi(1)-\chi(V_1)+\cdots+p^{R_i}\chi(1)-\chi(V_i)+\cdots]$$

is related to an invariant obtained from the defining equation of a totally ramified $K|k$, $e=[K:k]$. As usual V_i denote the ramification groups of $K|k$, p^{R_i} their orders, χ a character of the Galois group and

$$\chi(V_i)=\sum_{\sigma\in V_i}\chi(\sigma).$$

The crucial argument consists in interpreting the conductor by means of normal simple algebras $A|k$ of degree $[K:k]$ and index $\chi(1)$, χ a simple character of the abelian $K|k$. Let $\bar{K}$ be the subfield of K belonging to χ. All $\Gamma'\equiv\Gamma(\mathrm{mod}^{\times}\mathfrak{p}^E)$, Γ a prime of $\bar{K}$, $\Gamma'\in A$, $\mathfrak{p}$ the prime ideal of A, determine totally ramified fields $K'\cong\bar{K}\cong\bar{K}$. It is shown that E is maximal; finally, in terms of this interpretation it is immediate that E is integral. The proofs of the various statements are rather involved and can be simplified in parts.

O. F. G. Schilling (Chicago, Ill.).

Mac Lane, Saunders and Schilling, O. F. G. Infinite number fields with Noether ideal theories. Amer. J. Math. **61**, 771–782 (1939). [MF 38]

The authors prove that a field K with properties (i) every (exponential) valuation of K is discrete, and (ii) each element x of K has valuation $V(x)=0$ in only a finite number of non-equivalent valuations V of K, is either one of the classical number or function fields with finiteness properties, or is one of the new types: (I_∞) an infinite algebraic number field such that for each V of K there exists a subfield $K_V\subset K$, such that V is unramified and undecomposed in the extension from K_V to K; (II_∞) an infinite algebraic function field of one variable over an absolutely algebraic number field Ω of characteristic not equal to 0, such that K is separable over $\Omega(t)$ for a suitable t in K, and such that for every valuation V of K there is a finite algebraic function field $K_V\subset K$ with V unramified and undecomposed in the extension form K_V to K. To have property (i) alone delete "undecomposed" from the given conditions. There exists a field of type (I_∞) of degree m for every Steinitz G-number m, and there exist fields of type (II_∞) of certain degrees. The ring J of all algebraic integers of a field of type (I_∞) has a classical Noether ideal theory, that is, unique decomposition into prime ideal factors and $\mathfrak{a}\subset\mathfrak{b}$ implies $\mathfrak{a}=\mathfrak{bc}$, where $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ are ideals. [Contrast Krull, Math. Z. **29**, 42 (1928), for the case of infinite algebraic number fields in general.] Conversely, if a domain of integrity has the classical ideal decomposition theory, and if, in addition, all its elements are integral in all its valuations, then it is either an absolutely algebraic field of characteristic not equal to 0, or the ring of algebraic integers in an algebraic number field which is either finite or of type (I_∞). Certain transcendental rings in fields of type (II_∞) also have the classical ideal theory. *R. Hull* (Vancouver, B. C.).

THEORY OF NUMBERS

Pillai, S. S. On normal numbers. Proc. Indian Acad. Sci., Sect. A. **10**, 13–15 (1939). [MF 78]

The author considers, among others, the number $.123\cdots$ (in the scale r) formed by writing the positive integers (in the scale r) in succession. He gives a proof that these numbers are simply normal, that is, each digit from 0 to $r-1$ appears with the asymptotic frequency $1/r$. The proof of the stronger statement that these numbers are normal is inadequate. *H. S. Zuckerman* (Seattle, Wash.).

Vandiver, H. S. Certain congruences involving the Bernoulli numbers. Duke Math. J. **5**, 548–551 (1939). [MF 173]

As a complementary theorem to the relation known as Kummer's congruence [J. Reine Angew. Math. **41**, 368–372 (1851)] it is proved that, if p is an odd prime, then

$$b^{a(p-1)}(b^{p-1}-1)^j\equiv 0 \bmod p^{j-1},$$

where the left member is expanded in full, b_k is substituted for b^k, and $a>0$, $j>0$, $a+j<p-1$. Here the b_k are the Bernoulli numbers defined (for $n>1$) by the symbolic formula $(b+1)^n=b_n$. *R. D. Carmichael* (Urbana, Ill.).

Erdős, P. Note on products of consecutive integers. J. London Math. Soc. **14**, 194–198 (1939). [MF 242]

The purpose of this note is to prove that the product of k consecutive integers is never a perfect square. The proof makes use of the theorem (due to Sylvester) that such a product is divisible by a prime greater than k [Mess. of Math. **21** (1892); Sylvester's Collected Mathematical Papers **4**, 687]. The author has given an elementary proof of Sylvester's theorem [J. London Math. Soc. **9**, 282–288 (1934)]; the methods used in this note are again elementary and somewhat similar to those used in the paper just referred to. The proof is given for $k\geqq 100$; the remaining cases are taken care of by referring to a paper by S. Narumi who had already given the proof for $k\leqq 202$ [Tôhoku Math. J. **11**, 128–142 (1917)]. The author states that the same method can be used to prove that a product of consecutive odd integers is never a power. *H. W. Brinkmann.*

Ward, Morgan. Note on the general rational solution of the equation $ax^3-by^3=z^3$. Amer. J. Math. **61**, 788–790 (1939). [MF 40]

The author finds the general rational solution of

$$a_0x^m+a_1x^{m-1}y+\cdots+a_my^m=z^n,$$

where m and n are co-prime integers and $a_0, a_1, \cdots, a_m$ are rational, by reducing it to the form $Y^m=Z^n$. This last equation is a particular case of an equation previously discussed by the author [Amer. J. Math. **55**, 67–76 (1933); **59**, 921–926 (1937)]. The case $m=2$, $n=3$, $a_0=a$, $a_1=0$, $a_2=-b$, $a_3=a_4=\cdots=a_m=0$ leads to an equation solved by Fogels by the method of algebraic numbers [Amer. J. Math. **60**, 734–736 (1938)]. *I. A. Barnett* (Cincinnati, Ohio).

Gloden, A. Sur la résolution en nombres entiers du système $A^{kz}+B^{kz}+C^{kz}+D^{kz}=E^{z}+F^{z}+G^{z}+H^{z}$ $(x=1, 2$ **et** $3)$. Bol. Mat. **12**, 118–122 (1939). [MF 81]

Rosser, Barkley. On the first case of Fermat's last theorem. Bull. Amer. Math. Soc. **45**, 636–640 (1939). [MF 63]

Employing a theorem of Morishima [Jap. J. Math. **11**, 241–252 (1935)], the author proves that for no prime $p<8$, 332, 403 will the equation (1) $x^p+y^p+z^p=0$ have integral solutions x, y, z in the field of the pth roots of unity, with xyz prime to p. The method of proof consists in a comparison of the number of solutions x, $0<x<p^2/2$, of the congruence (2) $x^{p-1}\equiv 1 \pmod{p^2}$ with the number of numbers (M), within the same limits, generated by the first eleven primes. For, should the equation (1) possess solutions of the above mentioned kind for a given p, then by Morishima's theorem all numbers (M) satisfy congruence (2). Should, therefore, the number of numbers (M) between 0 and $p^2/2$ exceed the number of possible solutions of (2) within the same limits, then the corresponding equation (1) would have no solution. A lower bound $f_n(\log N)$ for the number $\phi_n(N)$ of positive integers not greater than N and having no prime factor greater than the nth prime, is determined in the paper and the case $n=11$ with $N=p^2/2$ is used in the comparison. *A. E. Ross* (St. Louis, Mo.).

Davenport, H. On Waring's problem for cubes. Acta Math. **71**, 123–143 (1939). [MF 213]

There are very few results concerning Waring's problem which may be called best possible. One of these is the main theorem of this paper: If $E_s(N)$ denotes the number of positive integers not greater than N which are not representable as a sum of s positive integral cubes, then $E_4(N)/N\to 0$ as $N\to\infty$. A simple argument shows that the theorem is false if $s<4$.

The new weapon used here has been explained in detail in several papers by the author [C. R. Acad. Sci. Paris **207**, 1366 (1938); Proc. Roy. Soc. London **170**, 293–299 (1939); Ann. of Math. **40**, 533–536 (1939)]. The remainder of the proof is similar to that given by Landau [Vorlesungen über Zahlentheorie, Bd. 1, 235–303] for the third Hardy-Littlewood theorem [Satz 346] as adapted by Davenport and Heilbronn [Proc. London Math. Soc. (2) **43**, 73–104 (1937)] to the problem of representations by two cubes and one square. Interesting parts of the proof are the use of Poisson's summation formula in Lemma 7 and the handling of the Singular Series. *R. D. James* (Saskatoon, Sask.).

Davenport, H. and Erdös, P. On sums of positive integral k**th powers.** Ann. of Math. **40**, 533–536 (1939). [MF 92]

In the application of the Hardy-Littlewood method to Waring's problem [Landau: Vorlesungen über Zahlentheorie, Bd. 1, Teil VI, Kap. 4] use is made of inequalities of the form $N_s^{(k)}(n)>n^{\alpha-\epsilon}$, $\alpha=\alpha(k,s)$. Here $N_s^{(k)}(n)$ denotes the number of integers $m\leqq n$ for which the equation $m=h_1^k+h_2^k+\cdots+h_s^k$ has a solution in integers $h_i\geqq 0$. The larger α can be made the better the final result. In a previous paper, Davenport [Proc. Roy. Soc. London, Ser. A. **170**, 293–299 (1939)] defined admissible exponents as follows: The real numbers $\lambda_1, \lambda_2, \cdots, \lambda_s$ satisfying $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_s>0$ are called admissible exponents for kth powers if the number of solutions of $x_1^k+\cdots+x_s^k=y_1^k+\cdots+y_s^k$ in integers $x_1, \cdots, x_s, y_1, \cdots, y_s$, subject to the conditions $P^{\lambda_i}<x_i<2P^{\lambda_i}$, $P^{\lambda_i}<y_i<2P^{\lambda_i}$, is $O(P^{\lambda_1+\cdots+\lambda_s+\epsilon})$ as $P\to\infty$ for any $\epsilon>0$. He then showed that if $\lambda_1, \cdots, \lambda_s$ are admissible exponents and $\alpha=(\lambda_1+\cdots+\lambda_s)/k\lambda_1$, then $N_s^{(k)}(n)>n^{\alpha-\epsilon}$, for any $\epsilon>0$ and $n>n_0(\epsilon)$. The present paper is concerned with the construction of admissible exponents for $k\geqq 3$ by the method of Erdös, with modifications due to Davenport. The following results are established: Theorem 1. Let $\theta=1-k^{-1}$. Then $1, \lambda, \lambda\theta, \cdots, \lambda\theta^{s-2}$ are admissible exponents for kth powers, provided λ satisfies $k\lambda-(k-1)\leqq\lambda\theta^{s-2}$. The proof is by induction on s. Theorem 2. Admissible exponents for kth powers are $1, 1-k^{-2}, 1-k^{-1}-k^{-2}$. The proof employs a device similar to the one used by Davenport for the case $k=4$ [C. R. Acad. Sci. Paris **207**, 1366 (1938)]. *R. D. James* (Saskatoon, Sask.).

Dickson, L. E. All integers except 23 and 239 are sums of eight cubes. Bull. Amer. Math. Soc. **45**, 588–591 (1939). [MF 56]

A method similar to that of W. S. Baer [Dissertation, Göttingen, 1913] shows that every integer beyond a stated limit is a C_8. The integers below this limit are treated by several lemmas, proved by the use of tables which are not published here, but are deposited in manuscript in the library of the University of Chicago. *W. A. Hurwitz.*

Scherk, Peter. Bemerkungen zu einer Note von Besicovitch. J. London Math. Soc. **14**, 185–192 (1939). [MF 241]

The note [J. London Math. Soc. **10**, 246–248 (1935)] to which reference is made in the title of this paper is concerned with the density of the sum of two sequences of integers. Suppose that A, B, C, and C' are four sequences of integers, where C is the sequence consisting of all elements of A and B and of all sums of an element of A and an element of B, and where C' is the sequence consisting of all elements of A and of all sums of an element of A and an element of B. Let $A(m,k), \cdots, C'(m,k)$ denote the number of integers in the respective sequences which exceed k but which do not exceed m. A simple argument shows that $A(m,k)+B(l-k-1, l-m-1)\leqq k-m$, where l is not an element of C and $0\leqq m<k<l$. From this result three inequalities for the function $C(n,0)$ are obtained which bound it from below. Similar inequalities hold for $C'(n,0)$. In every case sequences A and B are exhibited for which the sign of equality holds. Finally, with the aid of one of these inequalities the following theorem is proved: Suppose that $\alpha>0$, $\beta>0$, $\alpha+\beta<1$, $0\leqq\beta-\delta<1$. Suppose further that $A(m,0)\geqq\alpha m$, $B(m,0)\geqq\beta m+\delta$ for $m=1, 2, \cdots, n$. Then

$$C'(n,0)\geqq n\cdot\max\left\{\frac{\alpha+\delta}{1-\beta+\delta}, \frac{\alpha+\beta}{1+\beta-\delta}, \alpha+\beta-\alpha(\beta-\delta)\right\}.$$

Proofs of earlier results of which this theorem is a generalization may be found in Landau's Cambridge Tract: Über einige neuere Fortschritte der additiven Zahlentheorie. *R. D. James* (Saskatoon, Sask.).

THEORY OF GROUPS

Weyl, Hermann. Invariants. Duke Math. J. **5**, 489–502 (1939). [MF 167]

Invariants, covariants, and related notions are discussed from the point of view of modern theory of groups and representations. A short historical introduction is also given.

Hilbert's ω-process and its relationship to Cayley's algebraical Ω-process and to Hurwitz's and Weyl's analytical and Haar's set-theoretical integration processes are dis-

cussed, and connections with the theories of almost periodicity hinted. The rôles of the classical simple Lie groups and of the allied permutation groups are discussed.

Finally the place of the entire subject in modern abstract algebra, as well as its further possibilities under this aspect, is expounded. *J. von Neumann* (Princeton, N. J.).

Miller, G. A. Groups which contain ten or eleven proper subgroups. Proc. Nat. Acad. Sci. U. S. A. **25**, 540–543 (1939). [MF 270]

The author lists all groups which contain ten or eleven proper subgroups. In the former case there are eleven types, of which five are abelian. In the latter case, there is just the single type, which is cyclic of order p^{12}.

J. S. Frame (Providence, R. I.).

Weisner, Louis. Condition that a finite group be multiply isomorphic with each of its irreducible representations. Amer. J. Math. **61**, 709–712 (1939). [MF 28]

It may happen that all the irreducible representations of a finite group G are multiply isomorphic with G. The first study of this question was made by W. Burnside [The Theory of Groups of Finite Order, 2nd ed., Cambridge, 1911, 476] who did not, however, obtain necessary and sufficient conditions for this exceptional case. The first set of such conditions was given by K. Shoda [J. Fac. Sci. Univ. Tokyo, Sect. 1, 2, 51–72 (1930)]. Weisner gives conditions which are simpler than Shoda's. He proves: Each irreducible representation of G is multiply isomorphic with G, if and only if G contains a subgroup P of order greater than 1 such that (1) P is generated by the minimal invariant subgroups of G whose orders are powers of a fixed prime p; and (2) every subgroup of index p of P contains an invariant subgroup of G different from the identity group. *R. Brauer.*

Hopkins, Charles. An extension of a theorem of Remak. Ann. of Math. **40**, 636–638 (1939). [MF 103]

Let T denote a set of (proper) automorphisms of the group G, and suppose that T contains the inner automorphisms of G and that the T-admissible subgroups of G satisfy the minimal condition. Then there exists at least one minimal T-admissible subgroup different from 1, if only $G \neq 1$; and there exists therefore the subgroup S, generated by all the T-admissible minimal subgroups of G. S turns out to be a direct product of a finite number of minimal T-admissible subgroups, and these direct factors are uniquely determined, apart from central-isomorphisms. Let M be some minimal T-admissible subgroup of G, and suppose that the minimal condition holds for the normal subgroups of M. Then M is the direct product of a finite number of simple groups; and if M is not abelian, then these direct factors are uniquely determined and they form a complete set of subgroups which are conjugate under T. *R. Baer.*

Baer, Reinhold. Groups with Abelian norm quotient group. Amer. J. Math. **61**, 700–708 (1939). [MF 27]

The norm (Kern) $N(G)$ of a group G consists of all those elements u of G for which $uXu^{-1}=X$ for every subgroup X of G [cf. R. Baer, Compositio Math. **1**, 254–283 (1934)]. Obviously, $N(G)$ is a characteristic subgroup of G which contains the centre $C(G)$ of G. In a former paper [Compositio Math. **4**, 1–77 (1936)], Baer has studied those groups for which $G/N(G)$ is abelian and the norm of G is different from the center. In particular, he was interested in the extension problem of finding all possible groups G when $N(G)$ and the factor group $G/N(G)$ are given. Of special importance here, naturally, is the group $A(G)$ consisting of those automorphisms of $N(G)$ which are induced by elements of G. In the new paper, this work is continued. It is possible to restrict oneself to the case of a p-group (of finite or infinite order). Let $C(N(G)<G)$ be the centraliser of $N(G)$ in G, consisting of the elements c in G for which $cn=nc$ for every n in $N(G)$. The author proves that if $G/N(G)$ is abelian and $N(G) \neq C(G)$, and $p \neq 2, 3$, then there exists a maximal order p^r of the elements of $C(N(G)<G)$. This centraliser consists of exactly those elements in G whose p^rth power is the unit element. The factor group $N(G)/C(G)$ is cyclic; its order is the maximal order in the group $A(G)$. The group $A(G)$ is mapped isomorphically upon a subgroup of $C(G)$ by the correspondence $g \to g^{p^r}$. Finally, we have $g^{-1}xg=x(g^{p^r})^{h(x)}$ for every x in $N(G)$, g in G, where $h(x)$ is an integer which is independent of g. These statements appear as a special case of a more general theorem where $N(G)$ is replaced by a normal subgroup N of G which is contained in $N(G)$ but not in $C(G)$. Conditions are given under which the assumption $p \neq 2, 3$ can be dropped. *R. Brauer* (Toronto, Ont.).

Baer, Reinhold. Nets and groups. Trans. Amer. Math. Soc. **46**, 110–141 (1939). [MF 15]

A net consists of points and three families R, S, T of lines. Through every point there passes exactly one line of each family; if two lines belong to different families, then they have exactly one point in common.

Every net may be derived from a multiplicative system M, with unit, in which $ax=b$, $ya=b$ have unique solutions $x=a\backslash b$, $y=b/a$, in the following way. The points correspond to the pairs (x, y), the R, S, T lines have the equations $x=\text{const.}$, $xy=\text{const.}$, $y=\text{const.}$ Every such M, even without unit, determines a net.

In a similar fashion the author makes nets correspond to systems P of permutations, containing identity, which transform an object exactly once into any object. The systems P and P' determine isomorphic nets if they are similar, that is, if two (1-1) correspondences s, t exist such that $P=tP's^{-1}$. Two systems M and M' are similar, and yield isomorphic nets, if there exists a (1-1) correspondence $M \longleftrightarrow M'$ and two elements U, V such that

$$X'Y'=\{[(XV\backslash U)((V/U)\backslash(YV))]/V\}'.$$

If M is a group, then P is a group, similarity reduces to isomorphism, and the net is called a group net. In theorem 8.1.(2), the author gives a new characterization of group nets; however, the proof does not seem to be conclusive.

The theory of nets is preceded by a study of the relations between systems M and groups. Let S be a subgroup of G, and $r(X)$ a system of representatives for the right cosets Sx. Then $(Sr(X))(Sr(Y))=Sr(X)r(Y)$ makes the system of the cosets a left division system with left unit. A system M, where division is possible on both sides, is obtained if and only if the $r(X)$ represent also the cosets of the conjugates gSg^{-1}. Without loss of generality, one may reduce everything modulo the greatest invariant subgroup of G in S. If, therefore, the common part of S and all its conjugates is 1 (canonical representation), we have the theorems: (a) every system M has a canonical representation; (b) the $r(X)$ generate the group G; (c) the multiplication of the cosets determines uniquely the representatives $r(X)$; (d) there is a unit element exactly if 1 occurs as a representa-

tive; (e) the canonical representation is essentially unique; (f) all similar systems may be obtained by passing from $r(X)$ to $r(V)r^{-1}(U)r(X)r^{-1}(V)$.

The interrelations between nets, systems M, systems P, and groups are described very extensively. *M. Zorn.*

Hirsch, K. A. On skew-groups. Proc. London Math. Soc. **45**, 357–368 (1939). [MF 249]

By a skew-group, the author understands a group $\mathfrak{G}$ with the following properties: (1) $\mathfrak{G}$ has a finite number of generators, but does not contain elements of finite order, except the unit element 1; (2) the central $\mathfrak{Z}$ of $\mathfrak{G}$ is cyclical, $\mathfrak{Z}=\{Z\}$, and contains the derived group of $\mathfrak{G}$. It is shown that a set of elements $P_1, Q_1, \cdots, P_n, Q_n$ can be chosen such that (a) they form (mod $\mathfrak{Z}$) a set of independent generators of the free Abelian group $\mathfrak{G}/\mathfrak{Z}$ and, of course, the P_i, Q_i, and Z generate $\mathfrak{G}$; (b) there are positive integers $k_1, k_2, \cdots, k_n$ with $k_i \mid k_{i+1}$ such that $(P_i, Q_i)=P_i^{-1}Q_i^{-1}P_iQ_i=Z^{k_i}$, whereas all the other commutators $(P_i, Q_j)=1$, $(P_i, P_j)=1$, $(Q_i, Q_j)=1$. These statements are equivalent to the following one: If F is a nondegenerate skew-symmetric bilinear form in cogredient variables $(x_1, y_1, \cdots, x_n, y_n)$ and $(x_1', y_1', \cdots, x_n', y_n')$ with integral rational coefficients, then F can be transformed into a form $H=\sum k_i(x_iy_i'-x_i'y_i)$ by a unimodular transformation. The "elementary divisors" k_i characterize $\mathfrak{G}$ completely; and to any set k_i belongs a group $\mathfrak{G}$, if the k_i satisfy the given conditions. Further, the group $\mathfrak{A}$ of automorphisms of $\mathfrak{G}$ is studied. It is proved that $P_i \to \Pi_j P_j^{\alpha_{ij}}Q_j^{\beta_{ij}}$, $Q_i \to \Pi_j P_j^{\epsilon_{ij}}Q_j^{\delta_{ij}}$, $Z \to Z$ forms an automorphism, if and only if $x_i \to \sum a_{ij}x_j+\sum b_{ij}y_j$, $y_i \to \sum c_{ij}x_j+\sum d_{ij}y_j$ is a unimodular transformation which leaves the form H absolutely invariant. These automorphisms form a subgroup $\mathfrak{B}$ of $\mathfrak{A}$, for which generating elements are given. The most general automorphism is obtained by combining elements of $\mathfrak{B}$ with automorphisms of the forms $P_i \to P_iZ^{\alpha_i}$, $Q_i \to Q_iZ^{\alpha_i}$, $Z \to Z$ and $P_i \to Q_i$, $Q_i \to P_i$, $Z \to Z^{-1}$. If a group $\mathfrak{L}$ without elements of finite order has a finite number of generators, and if the derived group is cyclical and contained in the central of $\mathfrak{L}$, then $\mathfrak{L}$ is the direct product of a skew-group and an Abelian group. *R. Brauer.*

ANALYSIS

***Miller, Frederic H. Calculus.** John Wiley and Sons, Inc., New York, 1939. xiv+419 pp. $3.00.

Lorenzen, P. Die Definition durch vollständige Induktion. Monatsh. Math. Phys. **47**, 356–358 (1939). [MF 66]

The question of uniqueness of the sum of natural numbers under the Peano postulates requires demonstration as pointed out by Landau, but Dedekind had previously raised a similar issue. The author aims to avoid both the artificiality of a special summation symbol, and the dependence upon order relations, used by previous writers. He studies the properties of one-valued everywhere-defined functions on a set to this same set, and concludes the uniqueness of addition as a corollary of the general theory.

A. A. Bennett (Providence, R. I.).

Young, L. C. On an inequality of Marcel Riesz. Ann. of Math. **40**, 567–574 (1939). [MF 96]

Let $x=(x_1, \cdots, x_n)$ be a vector of dimension $n=2N$ and let

$$S_r(x)=\sum_{k=1}^{2N}|x_{2k-1}+ix_{2k}|^r, \qquad r>0.$$

Let A be an n-rowed nonsingular square matrix, and define

$$F_\alpha(x)=\{S_{1/\alpha}(x)\}^\alpha, \quad 0<\alpha\leqq 1, \quad F_0(x)=\lim_{\alpha\to 0}F_\alpha(x);$$

$$F_\beta^*(x)=F_\beta(x\cdot A), \quad 0\leqq\beta\leqq 1; \quad M_{\alpha\beta}=\max_x\{F_\beta^*(x)/F_\alpha(x)\}.$$

The author proves that $\log M_{\alpha\beta}$ is a convex function of (α, β) in the triangle $0\leqq\beta\leqq\alpha\leqq 1$; this is somewhat more general than the original theorem of Riesz [see Hardy, Littlewood, and Pólya: Inequalities, 1934, 219]. The proof depends on the fact that $F_\alpha(x)$ and $F_\beta^*(x)$ can be regarded as "functions of distance" (Distanzfunktionen) of convex bodies, and consequently satisfy an inequality established by the author, involving such functions and their transforms by means of a generalization of the transformation of reciprocal polars. If $F(x)$ and $\Phi(x)$ are functions of distance of convex bodies ($\Phi(x)$ differentiable), and $G(y)=\max_x\{(x\cdot y)/F(x)\}$ is the Stützfunktion corresponding to $F(x)$, the author defines $f(x)=\Phi(x)\{G(\partial\Phi/\partial x)\}^k$ as a transform of $F(x)$. If F^*, Φ^* are other functions of distance, the inequality in question is

$$\left\{\max_x\frac{f^*(x)}{f(x)}\right\}\left\{\max_x\frac{F^*(x)}{F(x)}\right\}^k\geqq\left\{\max_x\frac{\Phi^*(x)}{\Phi(x)}\right\}^{1+k}.$$

Riesz's inequality follows from this by use of the special elementary properties of the functions $F_\alpha(x)$ and $F_\beta^*(x)$.

R. P. Boas, Jr. (Durham, N. C.).

Mall, Josef. Ein Satz über die Konvergenz von Kettenbrüchen. Math. Z. **45**, 368–376 (1939). [MF 46]

The author considers the continued fraction

$$\text{(I)} \qquad \frac{1}{|b_1|}+\frac{1}{|b_2|}+\cdots,$$

the $b_{2\nu}$ all real and not equal to 0, $b_1\neq 0$, not all $b_{2\nu-1}=0$, and

$$b_{2\nu-1}=|b_{2\nu-1}|e^{i\theta_{2\nu-1}}, \quad 0<\epsilon\leqq\theta_{2\nu-1}\leqq\pi-\epsilon, \quad \text{if } b_{2\nu-1}\neq 0,$$

$$s_\nu=\sum_{n=1}^{\nu}b_{2n}\neq 0, \qquad \nu=1, 2, \cdots.$$

The author proves convergence of (I) if (i) one of the two series

$$\sum_{\nu=0}^{\infty}|b_{2\nu+1}|, \quad \sum_{\nu=1}^{\infty}s_\nu^2|b_{2\nu+1}|$$

diverges, (ii) both series converge and $\sum|b_{2n}|$ diverges. In all other cases (I) diverges.

This is a generalization of a result due to Hamburger [all $\theta_{2\nu-1}$ are equal, Math. Ann. **82**, 120–164 (1921)], and is proved here by means of fundamental formulae from the theory of continued fractions. *J. A. Shohat.*

Leighton, Walter and Scott, W. T. A general continued fraction expansion. Bull. Amer. Math. Soc. **45**, 596–605 (1939). [MF 58]

The authors establish 1-1 correspondence between power series $1+\sum_{n=1}^{\infty}c_nx^n$ and continued fractions of the type

$$1+\frac{a_1x^{\alpha_1}}{|1|}+\frac{a_2x^{\alpha_2}}{|1|}+\cdots$$

(x a complex variable, the a_i complex constants not equal to 0, the α_i positive integers), generalizing the notion of a

"corresponding" continued fraction (all $\alpha_i=1$, Perron). The case when the above continued fraction terminates is discussed, also its convergence and the nature of its limit-function. Finally, they discuss the existence of Perron's corresponding continued fraction for power series with gaps. *J. A. Shohat* (Philadelphia, Pa.).

Bonder, Julian. Über die Darstellung gewisser, in der Theorie der Flügelschwingungen auftretender Integrale durch Zylinderfunktionen. Z. Angew. Math. Mech. **19**, 251–252 (1939). [MF 145]

Theory of Sets, Theory of Functions of Real Variables

Kershner, Richard. The number of circles covering a set. Amer. J. Math. **61**, 665–671 (1939). [MF 23]

The author proves in a very simple manner that if M denotes a bounded plane point set, $\bar{M}$ its closure, and $N(\epsilon)$ the minimum number of circles of radius ϵ which can cover M, then

$$\lim_{\epsilon\to 0} \pi\epsilon^2 N(\epsilon) = (2\pi/3\sqrt{3}) \text{ meas. } \bar{M}.$$

P. Hartman (New York, N. Y.).

Dienes, Z. P. Canonic elements in the higher classes of Borel sets. J. London Math. Soc. **14**, 169–175 (1939). [MF 239]

Point sets considered are all in the interval (0, 1) and all sums and products mentioned are enumerable. A set is of type B_1 if both the set and its complement are sums of closed sets. A set is of type B_α if both the set and its complement are sums of products of sets of type less than α. The problem solved here is that of finding canonic elements for sets of type B_α. Canonic elements of the class B_α are sets of a few simple types belonging to B_α such that every B_α set is the sum of canonic elements of the class B_α and sets of type less than α. A set is of type R_α if it is of type B_β ($\beta\leqq\alpha$). In terms of canonic elements the author proves a necessary and sufficient condition that a set be of type R_α. *D. Montgomery* (Northampton, Mass.).

Marcinkiewicz, Józef and Zygmund, Antoni. Sur la dérivée seconde généralisée. Bull. Sém. Math. Univ. Wilno **2**, 35–40 (1939). [MF 115]

A function $f(x)$ is said to possess at the point $x=x_0$ a generalized second derivative with the (finite) value s, in the sense of de la Vallée Poussin, or, respectively, of Schwarz, if the condition

$$(1)\qquad \lim_{t\to 0} 2[f(x_0+t)-f(x_0)-at]/t^2 = s(x_0)$$

or

$$(2)\qquad \lim_{t\to 0} [f(x_0+t)+f(x_0-t)-2f(x_0)]/t^2 = s(x_0)$$

holds. Condition (1) implies (2) but not conversely. The authors have proved [Fund. math. **26**, 1–43 (1936)] that if (2) holds for $x_0\in E$, (1) holds almost everywhere in E.

In this note the condition (2) is still further generalized. Let $a<b<c$ be fixed; the generalized second derivative is said to exist and be equal to s, if

$$(3)\quad \lim_{t\to 0} [A\cdot f(x_0+at)+B\cdot f(x_0+bt)+C\cdot f(x_0+ct)]/t^2 = s(x_0),$$

where A, B, C are uniquely determined by linear relations involving a, b, c. Again (1) implies (3); the principal result is that, as in the case of (2), condition (1) must hold almost everywhere on any set E where (3) holds. *J. A. Clarkson.*

Kempisty, Stefan. Sur les fonctions à variation bornée au sens de Tonelli. Bull. Sém. Math. Univ. Wilno **2**, 13–21 (1939). [MF 113]

Following Radó, the area of a surface $z=F(x, y)$ can be expressed in terms of a Burkill integral, the integrand of which is formed by means of Geöcze's functions of intervals $G_i(F, R)$, $i=1, 2$ [cf. Saks: Theory of the Integral, 1937, chap. V]. Since the G_i are defined by means of a Riemann integral, this implies two passages to the limit. To avoid this, the author introduces the functions of intervals

$$\Lambda_1(F, R) = k\cdot\min_{b\leqq y\leqq b+k} |F(a+h, y)-F(a, y)|,$$

$$\Lambda_2(F, R) = h\cdot\min_{a\leqq x\leqq a+h} |F(x, b+k)-F(x, b)|,$$

where R stands for the interval $a\leqq x\leqq a+h$, $b\leqq y\leqq b+k$. It is shown that the upper Burkill integral of Λ_i coincides with the Burkill integral of G_i and that, starting from Λ_i, the area of $z=F(x, y)$ can be expressed as a Burkill integral. Using a known theorem [Saks, p. 179], it follows, furthermore, that a necessary and sufficient condition that $F(x, y)$ be of bounded variation in Tonelli's sense is that Λ_1 and Λ_2 be of finite variation. *W. Feller* (Providence, R. I.).

Radó, T. and Reichelderfer, P. On a stretching process for surfaces. Amer. J. Math. **61**, 645–650 (1939). [MF 21]

Let T: $\bar{\rho}=\rho\cdot\lambda(\rho; r, R)$, $\bar{\varphi}=\varphi$, $\bar{\theta}=\theta$ (ρ, φ, θ being spherical coordinates) be a transformation defined on the region G: $r\leqq\rho\leqq R$ in which $\lambda(r; r, R)=0$, $\lambda(R; r, R)=1$, and λ is monotone increasing and of class C' for $r\leqq\rho\leqq R$. Let S be a smooth surface in G, $\bar{S}$ be the surface corresponding to S under T, A and $\bar{A}$ be their respective areas, and let $U(T; r, R)$ be the least upper bound of $\bar{A}/A$ for all such S in G. In a paper: An analytic characterization of surfaces of finite Lebesgue area [Amer. J. Math. **58**, 312–322 (1936), particularly p. 315], C. B. Morrey uses such a transformation $T(r, R)$ and obtains the bound $U(T; r, R)=R/(R-r)$, a bound which has the property, essential to the results of that paper, that $U(T; R, r)\to 1$ and $r/R\to 0$. The present paper shows that the above bound is incorrect, the correct bound for the T used being $2+(r/(R-r))$. The present paper fills this gap completely by (1) showing that the greatest lower bound of U for all T is $R^2/(R^2-r^2)$ and (2) finding simple transformations T for which U is arbitrarily near this bound. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Theory of Functions of Complex Variables

Ulrich, F. E. The problem of type for a certain class of Riemann surfaces. Duke Math. J. **5**, 567–589 (1939). [MF 175]

A criterion of Ahlfors [C. R. Acad. Sci. Paris **201**, 30–32 (1935)] states that a surface is parabolic if a metric $ds=\lambda|dw|$ exists such that $\int^\infty d\rho/L(\rho)=\infty$, where $L(\rho)$ is the length of the curve whose points are at a distance ρ from an initial point. This test is applied to a class of surfaces obtained by deformation of the surface of the Γ-function; the algebraic branch-points of this surface are moved to the points A_ν ($\nu=1, 2, \cdots$) on the real axis, $A_\nu>0$ for ν odd, $A_\nu<0$ for ν even. A sufficient condition that the surface be of parabolic type is the divergence of the series

$$\sum 1/\nu^2 \log c_\nu/|A_1|,$$

$c_\nu=2|A_1|+\cdots+2|A_{\nu-1}|+|A_\nu|$. *L. V. Ahlfors.*

Teichmüller, Oswald. Erreichbare Randpunkte. Deutsche Math. **4**, 455–461 (1939). [MF 132]

The author discusses two definitions of accessible bound-

ary points of a Riemann surface. In one of the definitions, in particular, the author obtains the accessible boundary points by closing the open surface with respect to a certain metric. The equivalence of the two definitions is proved. One of the purposes of this paper is to correct some errors that were found in L. Bieberbach's treatment of the subject in his Lehrbuch der Funktionentheorie, vol. 2, 1st edition, 1927. The paper is concluded with a number of corrections of Bieberbach's book. *W. Seidel* (Rochester, N. Y.).

Denjoy, Arnaud. Sur certaines séries de Taylor admettant leur cercle de convergence comme coupure essentielle. C. R. Acad. Sci. Paris **209**, 373–374 (1939). [MF 236]

The author constructs a power series $c_0+c_1z+c_2z^2+\cdots$, which converges for $|z|<1$ and represents an analytic function whose only singularities in the entire z plane are a perfect set of points of measure zero on $|z|=1$. On the other hand $\sum|c_n|^2z^n$ and other similar series have the unit circle as a natural boundary. *N. Levinson.*

Robertson, M. S. The variation of the sign of V for an analytic function $U+iV$. Duke Math. J. **5**, 512–519 (1939). [MF 169]

Let

$$f(z)=\sum_{n=0}^{\infty}a_nz^n=U+iV$$

be a regular analytic function in $|z|<1$. Let there be an interval $0<1-\delta<r<1$ such that V does not change sign more than $2p$ times on $|z|=r$ for any value of r of the given interval. Setting

$$\mu_p=\max\{|a_1|, |a_2|, \cdots, |a_p|\},$$

the author proves the following inequalities:

$$|a_n|<A(p)\mu_pn^{2p}, \qquad n>0;\ p\geqq 0,$$

$$\limsup_{n\to\infty}\left|\frac{a_n}{n^{2p}}\right|\leqq 2\sum_{k=0}^{p}\frac{|a_k|}{(p+k)!(p-k)!},$$

$$|f(re^{i\theta})|<A(p)\mu_p(1-r)^{-2p-1}, \qquad r<1,$$

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta<A(p)\mu_p(1-r)^{-2p}, \quad p\geqq 1;\ r<1,$$

where $A(p)$ is a constant depending on p alone. If, in addition, $f(z)$ is assumed to be real on the real axis, the author finds sharper inequalities. *W. Seidel* (Rochester, N. Y.).

Zygmund, Antoni. Sur un théorème de M. Fejér. Bull. Sém. Math. Univ. Wilno **2**, 3–12 (1939). [MF 112]

Fejér has shown [see, for example, Landau: Darstellung und Begründung, Berlin, 1929] that if

(1) $$f(z)=a_1z+a_2z^2+\cdots$$

is a function regular in the unit circle $|z|<1$ and such that

(2) $$\int_0^1\rho\,d\rho\int_0^{2\pi}|f'(\rho e^{i\theta})|^2d\theta<\infty,$$

then the series (1) converges almost everywhere on the circumference $|z|=1$. If, besides, the function $f(z)$ is continuous on a closed arc of the circumference $|z|=1$, the series (1) converges uniformly on this arc.

The author gives two demonstrations that Fejér's theorem can be localized. More precisely, let Γ be a domain contained in the unit circle and bounded by a Jordan curve having the arc $z=e^{i\theta}$, $\alpha\leqq\theta\leqq\beta$, common with the circumference $|z|=1$. If $a_n=o(1)$ and if

$$\iint_\Gamma|f'(\rho e^{i\theta})|^2\rho\,d\rho\,d\theta<\infty,$$

then the series (1) converges almost everywhere on the arc $z=e^{i\theta}$, $\alpha\leqq\theta\leqq\beta$. If, further, the function $f(z)$ is continuous at each point interior or on the boundary of Γ, the series (1) converges uniformly on each arc (α',β') interior to (α,β). The author makes use of the theory developed by Rajchman [Math. Ann. **95**, 388–408 (1926)] and Zygmund [Math. Z. **24**, 47–104 (1926)] on the formal multiplication of trigonometric series.

The theorem is extended also to the case where $a_n=o(n^\gamma)$, $\gamma\geqq 0$, replaces $a_n=o(1)$. The theorem is then still true if one also substitutes "is summable (C,γ)" for the word "converges." *M. S. Robertson* (New Brunswick, N. J.).

Zorn, Max. Continuous groups and Schwarz' lemma. Trans. Amer. Math. Soc. **46**, 1–22 (1939). [MF 9]

This is a contribution to the theory of conformal mappings from a topological point of view. The author considers a family N of single-valued continuous mappings of a metric space S into itself; N is subjected to restrictions which describe some of the topological properties of the special case in which S is the interior of the unit circle in the complex plane and N the totality of analytic (conformal) mappings F $(F(S)\subset S)$. The decisive restriction on N is that of normality: an infinite sequence $\{F_i\}$ of mappings in N either determines one or more limit mappings in N or else $\{F_i(x)\}$ has no limit point for any x. An F in N which admits a fixed point and has an inverse in N is called a rotation. The theory of rotations and "circumferences" about an arbitrary but definitely fixed point p is developed. For the definition of "circle" (circular region) further restrictions on N, S stating certain properties of separation are introduced. It then becomes possible in terms of abstract circles and circumferences to define relations of the form $|x|\leqq|y|$ which are equivalent in the case of analytic mappings to inequalities between moduli. A sequence of theorems leads to the following result: If $F(p)=p$ $(F\subset N)$, then $|F(x)|\leqq x$ for every x. This is the formulation, appropriate to the general setting under consideration, of Schwarz' lemma about analytic mappings. In the last section it is shown that every S which forms part of a system S, N must be separable, locally compact, and semi-compact. *P. A. Smith* (New York, N. Y.).

Levi, Howard. On the values assumed by polynomials. Bull. Amer. Math. Soc. **45**, 570–575 (1939). [MF 53]

If $f(x)$ is a polynomial of degree $n+1$ and k is given, the equation

$$\frac{f(x+h)-f(x)}{h}=k$$

has n solutions $h=h_i(x,k)$, $i=1,2,\cdots,n$, for which obviously

$$\frac{\partial}{\partial x}\sum_{i=1}^{n}h_i=-(n+1).$$

The converse of this remark is proved under the following assumptions: $f(x)$ is an integral function, h_i is analytic if x and k are restricted to certain domains of the complex x- and k-plane, respectively, $h_i\neq h_j$ if $i\neq j$, $h_i\neq 0$. An extension to polynomials of N variables is also obtained.

G. Szegö (Stanford University, Calif.).

Walsh, J. L. On interpolation by functions analytic and bounded in a given region. Trans. Amer. Math. Soc. **46**, 46–65 (1939). [MF 11]

The following set of "interpolation points β_{nk}" is defined. Let the Jordan curve C_1 contain the Jordan curve C_2 and let β_{nk} lie on C_2. Assuming that

$$\lim_{n\to\infty} |(z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{nn})|^{1/n}=e^{V_1(z)}$$

exists, uniformly in every closed domain exterior to C_2, let $V_2(z)$ be harmonic in C_1 and $V_2=V_1$ on C_1; also, V_1-V_2 should be constant on C_2. This is a generalization of certain older considerations due to Fejér, Kalmár and the author [see the book of the latter in the Colloquium Publications].

Let $f(z)$ be analytic in the closed interior of a level-curve C_λ of V_1-V_2 (between C_1 and C_2), and let $f_n(z)$ be the analytic function for which

$$f_n(\beta_{nk})=f(\beta_{nk}), \qquad k=1, 2, \cdots, n,$$

and $\max |f_n(z)|$ on C_λ is a minimum. Then $\lim_{n\to\infty} f_n(z)=f(z)$, uniformly in the interior of C_λ. Bounds for the difference $|f(z)-f_n(z)|$ on C_λ are also given. In the further part of the paper a form for this theorem is obtained which is invariant under conformal transformation. Generalizations of various kinds are indicated. *G. Szegö.*

Wirtinger, Wilhelm. Über eine Minimalaufgabe im Gebiete der analytischen Funktionen von mehreren Veränderlichen. Monatsh. Math. Phys. **47**, 426–431 (1939). [MF 71]

As a partial extension of a previous paper in one complex variable [Wirtinger, Monatsh. Math. Phys. **29**, 377ff. (1932)] the author considers the following minimal problem: Given a continuous function Φ of $2n$ real variables in a hypersphere H, to find an analytic function f of n complex variables such that $\int|\Phi-f|^2d\omega=\min.$, where the integral is taken over the interior of H and $d\omega$ is the volume element. He not only shows the existence and uniqueness of such a function f but he also obtains an explicit formula for f in terms of an integral over H. We note that the same problem for biharmonic functions (real part of an analytic function of 2 complex variables) over general four-dimensional regions has been solved by Bergmann [Math. Ann. **109**, 324–348 (1934), especially p. 331; Compositio Math. **3**, 136–173 (1936), especially p. 144] with the aid of the theory of complex orthogonal functions and the kernel of the region. *W. T. Martin* (Cambridge, Mass.).

Bergmann, S. et Marcinkiewicz, J. Sur les fonctions analytiques de deux variables complexes. Fund. Math. **33**, 75–94 (1939). [MF 164]

The authors obtain results of the Fatou type for functions analytic in the special class of regions having a distinguished surface, that is, regions such that their (3 dimensional) boundary possesses a (2 dimensional) surface which plays a role analogous to that of the boundary of a region in one complex variable. [See Bergmann, Jber. Deutsch. Math. Verein. **44**, 238–252 (1933).] A region of this sort is the region D whose boundary is given by $[z_1=h(z_2, \lambda), |z_2|\leqq 1, 0\leqq\lambda\leqq 2\pi]$ and $[z_1=sh(z_2, \lambda), z_2=e^{i\varphi}, 0\leqq\lambda\leqq 2\pi, 0\leqq s\leqq 1]$, where $h(z_2, \lambda)$ is a given function analytic in z_2 for fixed λ and once differentiable in λ for fixed z_2. (The function h is subjected to certain other conditions relating to the geometric character of the boundary.) The distinguished surface S for D is $[z_1=h(z_2, \lambda), z_2=e^{i\varphi}, 0\leqq\lambda, \varphi\leqq 2\pi]$. Let $F(z_1, z_2)\in H_p$, that is, let F be analytic in D and let $F(s_1h(e^{i\varphi}, \lambda), s_2e^{i\varphi})$ belong to $L^p(p>1)$ as a function of λ and φ, uniformly for $0\leqq s_i\leqq 1$. Then they show that $\lim F(z_1, z_2)$ exists almost everywhere for a very general method of approach of (z_1, z_2) in D to values $(h(e^{i\varphi}, \lambda), e^{i\varphi})$ on S, and the limit function $f(\lambda, \varphi)\in L^p$. They also prove a theorem of the converse type going from a boundary function $f(\lambda, \varphi)\in L^p$ to a function $F\in H_p$ and such that $\lim \Re\{F(z_1, z_2)\}=f(\lambda, \varphi)$ almost everywhere. They prove these results by considering D as the sum of regions $D_{s_1s_2}$ each having a distinguished surface $S_{s_1s_2}\subset D$ and each being the pseudo-conformal map of a bicylinder. In this way they can apply certain maximal evaluations of the theory of one variable and by iteration pass to the results indicated. *W. T. Martin* (Cambridge, Mass.).

Theory of Series

Epstein, Leo F. A function related to the series for e^{e^z}. J. Math. Phys. Mass. Inst. Tech. **18**, 153–173 (1939). [MF 190]

The numbers K_n defined by the expansion

$$\exp(e^z)=e\cdot\sum_0^\infty \frac{K_n}{n!}x^n$$

have been studied by various authors. In this paper further properties of these integers are discussed and a generalization to non-integral n given. The paper contains some interesting formulae and tables. *O. Szász.*

Chand, Hukam. On some generalizations of Cauchy's condensation and integral tests. Amer. Math. Monthly **46**, 338–341 (1939). [MF 158]

The following two theorems are proved:

If $\phi(x)\downarrow 0$ as $x\to\infty$ and $F(x)=a^x\phi(a^x)$, then $\sum F(n)$ and $\int F(x)dx$ converge and diverge together.

If (i) $\phi(x)\downarrow 0$ as $x\to\infty$, (ii) $g(n)>0$ is monotonic increasing to infinity with n, (iii) the sequence (g_n-g_{n-1}) is either bounded or $(g_{n-1}-g_n)\leqq\lambda(g_n-g_{n-1})$ for every n, where λ is constant, (iv) $h(x)$ is, for $x\geqq 1$, a monotonic increasing function such that $h(n)=g(n)$ for every n and $h'(x)$ exists, then $\sum(g_{n+1}-g_n)\phi(g_n)$, $\sum(g_{n+1}-g_n)\phi(g_{n+1})$, $\int\phi(h(x))h'(x)dx$ converge and diverge together. *T. Fort.*

Witt, G. Zur Berechnung der Koeffizienten der Störungsfunktion. Astr. Nachr. **269**, 74–80 (1939). [MF 138]

The hypergeometric series commonly used in the numerical evaluation of the coefficients $B_s^{(i)}$ in

$$(1+\alpha^2-2\alpha\cos\varphi)^{-s}=\tfrac{1}{2}B_s^{(0)}+\sum_{i=1}^{\infty} B_s^{(i)}\cos i\varphi,$$

$$s=\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots; \quad 0<\alpha<1,$$

are power series in α^2 or $-\alpha^2/(1-\alpha^2)$. The author demonstrates that the slow convergence that renders these series unsuitable in many particular cases can be overcome by a transformation to hypergeometric series in t^4, t being defined by $t=\tan\frac{1}{4}\theta$, $\alpha=\sin\theta$, and by using series in λ, where $\exp(-\lambda)=1-t^4$. In some cases it is preferable first to express the original series with the aid of two different hypergeometric series. If α is very close to unity, power series in $(1-\alpha)^2/(1+\alpha)^2$, introduced by Gauss, are recommended. *D. Brouwer* (New Haven, Conn.).

Agnew, Ralph Palmer. On oscillations of real sequences and of their transforms by square matrices. Amer. J. Math. **61**, 683–699 (1939). [MF 26]

Given a square matrix $A=(a_{nk})$ and the transformation $y_n=\Sigma_k a_{nk}x_k$, it is required to find conditions in order that

$\Omega(y) \leqq \Omega(x)$ for every real bounded divergent sequence $\{x_n\}$. Here the oscillation $\Omega(s)$ of the sequence $\{s_n\}$ is defined as $\Omega(s) = \limsup_{m,n\to\infty} |s_m - s_n|$. It turns out that $A = A' + A''$, where both matrices are convergence-preserving, the elements of A' are constant by columns and A'' is multiplicative and oscillation-reducing for real bounded sequences. The matrix A'' is said to be multiplicative if there exists a fixed ρ such that $\lim y_n'' = \rho \lim x_n$ whenever the latter limit exists, where $y_n'' = \Sigma_k a_{nk}'' x_k$. The author gives a number of necessary and sufficient conditions in order that a multiplicative matrix A shall be oscillation-reducing for real bounded sequences. One of these is $\lim\sup_{n\to\infty}$ l.u.b. $|\Sigma_k \pm a_{nk}| \leqq 1$, where the l.u.b. refers to all possible choices of the signs. Similar questions are solved for norms of sequences, where the norm of $\{s_n\}$ is $L(s) = \limsup_{n\to\infty} |s_n|$, and for sequence-to-function transformations. *E. Hille.*

Garabedian, H. L. Hausdorff matrices. Amer. Math. Monthly **46**, 390–410 (1939). [MF 201]

The object of this article is to present the principal ideas of Hausdorff's memoir [Summationsmethoden und Momentfolgen, I and II, Math. Z. **9**, 74–109, 280–299 (1921)] in a form which should be readily intelligible to the student of analysis. *Extract from the introduction.*

Garabedian, H. L. A sufficient condition for Cesàro summability. Bull. Amer. Math. Soc. **45**, 592–596 (1939). [MF 57]

If k is a positive integer and $\Delta^{k-1}a_0 \neq 0$, while $\Delta^i a_0 = 0$ when $i \geqq k$, then the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is not summable $(C, k-1)$ but is summable (C, k) to the value $\sum_{n=0}^{k-1} \Delta^n a_0 / 2^{n+1}$.

R. P. Agnew (Ithaca, N. Y.).

Karamata, J. Über einen Tauberschen Satz im Dreikörperproblem. Amer. J. Math. **61**, 769–770 (1939). [MF 37]

The author, inspired by R. P. Boas' note with similar title on pages 161–164 of the same volume, tries to find a convergence condition which would include that of Boas. His condition reads: From

$$\frac{1}{x}\int_0^x s(t)dt \to s \quad \text{follows} \quad s(x) \to s \quad \text{as} \quad x \to \infty,$$

provided there exists a function $\phi(x)$, continuous and strictly increasing in $-\infty < x < \infty$, such that

$$\liminf_{x\to\infty} \min_{x \leqq y \leqq \lambda x} [\phi(s(y)) - \phi(s(x))] \geqq -w(\lambda) \to 0 \quad \text{as} \quad \lambda \to 1.$$

A special choice of $\phi(x)$ gives Boas' condition, even in a slightly more general form. *E. Hille* (New Haven, Conn.).

Meyer-König, Werner. Limitierungsumkehrsätze mit Lükkenbedingungen. I. Math. Z. **45**, 447–478 (1939). [MF 50]

This paper is concerned with Tauberian theorems of types closely related to Hardy's "high indices" theorem, but instead of dealing with types of summability of infinite reference, as is the case in the Hardy theorem, the author confines his attention to Cesàro and Hölder summability and closely related forms. Theorem 1 asserts that if for the sequence of indices n_ν, n_ν' $(\nu = 0, 1, \cdots)$, $n_\nu < n_\nu' < n_{\nu+1}$; $n_\nu' - n_\nu > \delta n_\nu$ $(\delta > 0)$, and if for the sequence s_n, $n|a_n| < M$ $(M > 0, n_\nu + 1 \leqq n \leqq n_\nu')$ and if, when $n_{\nu+1}' < n_{\nu+1}$, $|s_n| < M$ $(n_{\nu+1}' \leqq n \leqq n_{\nu+1} - 1)$, then it follows from $C_k - \lim s_n = s$, where k is a positive integer, that $\lim_{\nu\to\infty} s_{k_\nu} = s$, where $k_\nu = k_\nu$ (n_ν, n_ν'). Theorem 2 leaves off the last clause in the hypothesis and concludes from $h_{k_\nu}^{(k)} \to s$ that $s_{k_\nu} \to s$, where $h_n^{(k)}$ is the sequence of kth Hölder means. Theorem 3 is a related one-sided theorem. Later theorems concern Hölder summability of infinite order. *N. Wiener.*

Fourier Series and Integrals, Theory of Approximation

Zygmund, Antoni. Note on the formal multiplication of trigonometrical series. Bull. Sém. Math. Univ. Wilno **2**, 52–56 (1939). [MF 117]

Given two trigonometric series

$$S = \sum_{-\infty}^{\infty} c_n e^{inx}, \quad T = \sum_{-\infty}^{\infty} \gamma_n e^{inx},$$

their formal product ST is the series

$$\sum_{-\infty}^{\infty} C_n e^{inx}, \quad \text{where} \quad C_n = \sum_{p+q=n} c_p \gamma_q.$$

It is well known that, if $c_n > 0$, $\sum_{-\infty}^{\infty} |n|\,|\gamma_n| < \infty$, and if the series T is the Fourier series of a function $\lambda(x)$ such that $\lambda(x_0) = 0$, then the formal product ST converges to zero at x_0. These conditions may be relaxed but not essentially.

The author shows that when the function $\lambda(x)$ behaves very well in an interval (a, b), then it is possible to relax the conditions substantially. Thus, if $c_n \to 0$, $\sum |\gamma_n| < \infty$, and if $\lambda(x)$ vanishes in (a, b), then the formal product converges to zero at all inner points of (a, b).

The author extends these results to conjugate series and summable series and concludes by establishing a theorem analogous to that quoted for Cauchy products of ordinary series. *A. C. Offord* (Cambridge, England).

Marcinkiewicz, J. Sur la sommabilité forte de séries de Fourier. J. London Math. Soc. **14**, 162–168 (1939). [MF 238]

The author answers a question left open by Hardy and Littlewood [Fund. Math. **25**, 187(2) (1935)]. A series $\sum u_n$ is said to be H_2 summable if there exists a finite number s such that

$$\sum_{\nu=0}^{n} (s_\nu - s)^2 = o(n), \quad s_\nu = u_0 + u_1 + \cdots + u_\nu.$$

With this definition the author proves that, if $f(x) \in L$ over $(-\pi, \pi)$, then the Fourier series of $f(x)$

$$\frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

and its conjugate

$$\sum_{\nu=1}^{\infty} (b_\nu \cos \nu x - a_\nu \sin \nu x)$$

are both H_2 summable almost everywhere.

J. D. Tamarkin (Providence, R. I.).

McEwen, W. H. On the simultaneous approximation of a function and its derivatives by sums of Birkhoff type. Bull. Amer. Math. Soc. **45**, 576–582 (1939). [MF 54]

If $B_N(x)$ is the partial sum of the expansion of a given function $f(x)$ in series of characteristic solutions of the system consisting of a differential equation $L(u) + \lambda u = 0$ of the nth order and a set of n regular boundary conditions of the type considered by Birkhoff in his thesis [Trans. Amer. Math. Soc. **9**, 373–395 (1908)], $B_N^{(k)}(x)$ is shown to converge uniformly toward $f^{(k)}(x)$ on the interval for $k = 0, 1, \cdots, n-1$, provided that $f(x)$ satisfies the boundary conditions and $L(f)$ is integrable. The conclusion holds for $k = 0, 1, \cdots, 2n-1$ if f and $L(f)$ satisfy the boundary conditions and $L(L(f))$ is integrable, and similarly for deriva-

tives of still higher order. If $f(x)$ satisfies the boundary conditions and if $L(f)$ has a continuous qth derivative of limited variation, while $L(f)$ itself and its first $q-1$ derivatives vanish at the ends of the interval, the error $f^{(k)}-B_N^{(k)}$ is less than a constant multiple of N^{-q} for $k=0, 1, \cdots, n$. There is a corresponding theorem on degree of convergence for higher derivatives. *D. Jackson.*

Hartman, Philip. Mean motions and almost periodic functions. Trans. Amer. Math. Soc. **46**, 66–81 (1939). [MF 12]

In recent years several interesting contributions to the problem of mean motion of exponential polynomials and almost periodic functions have appeared. In the present paper the author discusses the mean motion in the case of an analytic almost periodic function $f(s)=f(\sigma+it)$ in a strip $\alpha<\sigma<\beta$ when considered on vertical lines $\sigma=\sigma_0$. By the mean motion $\mu(\sigma)$ of the function $F_\sigma(t)=f(\sigma+it)$ is understood the limit

$$\mu(\sigma)=\lim_{\tau\to\infty}\frac{\arg F_\sigma(\tau)}{2\pi\tau},$$

where $\arg F_\sigma(t)$ denotes the continuous branch of the argument of $F_\sigma(t)$ (determined mod π) for which $0\leqq\arg F_\sigma(0)<\pi$, provided this limit exists. Using the connection between the variation of the argument of an analytic function along a closed curve and the number of zeros inside the curve, the author bases the investigation on previous results due to Jessen [Math. Ann. **108** (1933)] concerning the average frequency of the zeros of an analytic almost periodic function $f(s)$ in a strip $\alpha'<\sigma<\beta'$. As shown by Jessen, this average frequency is closely connected with the function

$$\psi(\sigma)=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \log|f(\sigma+it)|\,dt,$$

which is a continuous convex function of σ. By modification of Jessen's argument, the author proves that the mean motion $\mu(\sigma)$ exists and is equal to $\psi'(\sigma)/2\pi$ for every σ for which the derivative $\psi'(\sigma)$ exists. He then proceeds to discuss more general conditions for the existence of the mean motion $\mu(\sigma)$ in the case where the Dirichlet exponents of the function $f(s)$ possess an integral basis $\{2\pi\lambda_1, 2\pi\lambda_2, \cdots\}$, and more particularly the case where this basis is finite. In the usual manner this leads to a representation of $f(s)$ in the form $F_\sigma(t)=Z(\lambda_1 t, \lambda_2 t, \cdots; \sigma)$, where $Z(\theta_1, \theta_2, \cdots; \sigma)$ is a function on the finite or infinite dimensional θ-torus, depending on the parameter σ. This enables him to prove the existence of $\mu(\sigma)$ under suitable conditions concerning sets of points where $Z(\theta_1, \theta_2, \cdots; \sigma)=0$; in the case of a finite basis conditions regarding the dimension of such sets play an important role. In the particular case of an exponential polynomial

$$f(s)=\sum_{k=1}^{n}a_k\exp(2\pi\Lambda_k s+i\alpha_k),$$

this leads to the result that $f(s)$ possesses a mean motion $\mu(\sigma)$ for all σ with the possible exception of a finite set. In the case of linearly independent exponents this exceptional set is shown to be empty in accordance with a classical result of Weyl [Enseignement Math. **16** (1914) and Amer. J. Math. **60** (1938)]. The paper ends with results regarding the smoothness properties of $\mu(\sigma)$ and some indications regarding mean motions of the Riemann ζ-function. It may be added that quite recently both Jessen [C. R. Acad. Sci. Paris **207** (1938)] and Weyl [Amer. J. Math. **61** (1939)] have arrived at the general result that $\mu(\sigma)$ exists for all σ in the case of any exponential polynomial $f(s)$. *H. Bohr* (Copenhagen).

Wecken, Franz. Abstrakte Integrale und fastperiodische Funktionen. Math. Z. **45**, 377–404 (1939). [MF 47]

The author shows that a modified form of abstract integral-theory can be applied to the discussion of almost-periodic functions. He bases metric theory not on a Borel corpus (σ-corpus) but on an algebraic system ("Σ-system"), defined as a set $\mathfrak{B}$ of elements $A, B, \cdots$ with the properties:

(1) $\mathfrak{B}$ is a Boolean algebra with the operations $A\dot{+}B$ (composition), AB (intersection), A' (complement), with the unity-element E and the null-element N, and with the order-relation $A\subset B$ (that is, $AB=A$);

(2) $\sum_{n=1}^{\infty}\dot{+}A_n$ exists and is unique;

(3) A completely additive measure function μA exists with the properties: $0\leqq\mu A\leqq\infty$; $\mu A=0$ only for $A=N$; $E_1, E_2, \cdots$ exist such that $\mu E_n<\infty$ $(n=1, 2, \cdots)$, $\sum_{n=1}^{\infty}\dot{+}E_n=E$.

If now for $-\infty<\lambda<\infty$ a function $E(\lambda)$ is defined with values in $\mathfrak{B}$ and with the properties $E(\lambda)\subset E(\mu)$ for $\lambda<\nu$; $E(\lambda)\to E(\nu)$ for $\lambda>\nu$, $\lambda\to\nu$; $E(\lambda)\to N$ as $\lambda\to-\infty$, $E(\lambda)\to E$ as $\lambda\to\infty$, then a "measurable real function f over $\mathfrak{B}$" is defined by the symbolic equation $E[f\leqq\lambda]=E(\lambda)$ and the integral $\int_E f d\mu$ is defined as $\int_{-\infty}^{\infty}\lambda d\sigma(\lambda)$, where $\sigma(\lambda)=\mu E(\lambda)$. The extension to complex f is immediate and function-spaces $\mathfrak{M}, \mathfrak{M}^p$ are set up in the usual manner. They are separable if and only if $\mathfrak{B}$ is separable.

For the application to almost-periodic functions, it is shown that certain classes of point-sets on the real axis, with appropriate association-laws and with Bohr-Besicovitch density [Acta Math. **57**, 203–292 (1931)] as measure-function, form a Σ-system. The general theory can therefore be applied to obtain (non-separable) function-spaces $\mathfrak{M}, \mathfrak{M}^p$ $(p\geqq1)$. These are then given a realization by families of concrete functions $f(x)$, and their close connection with the function-families of Bohr and Besicovitch [loc. cit.] and Kovanko [Ann. Mat. Pura Appl. (4) **9**, 1–24 (1931)] is established. In particular, the most general of these families is shown to be identical with that of the Kovanko α-functions.

A brief account is given of the connection with operator-theory in the non-separable space $\mathfrak{R}=\mathfrak{M}^2$ of quadratically integrable functions; $\mathfrak{M}$ is isomorphic with a ring of normal operators in $\mathfrak{R}$, and it is indicated how this viewpoint leads to generalizations of results of A. Wintner [Math. Z. **36**, 618–629 (1933)] on statistical independence of distribution-functions of almost-periodic functions. *E. H. Linfoot.*

McLachlan, N. W. Application of the Mellin inversion theorem to impulses. Math. Gaz. **23**, 270–277 (1939). [MF 209]

In this paper the author uses the p-multiplied form of the Laplace transformation and calls its inversion under the conditions of Jordan's Lemma "the Mellin Inversion Theorem." Pulses of finite length and those whose lengths approach 0 are called "impulses." Formulas relating the transforms of both types of "impulses," single and repeated, are developed formally.

The well-known solution of linear constant-coefficient ordinary integro-differential equations and systems of these by the Laplace transformation is indicated. Several elementary applied problems are partially solved.

J. L. Barnes (Merchantville, N. J.).

Schaeffer, A. C. The Fourier-Stieltjes coefficients of a function of bounded variation. Amer. J. Math. **61**, 934–940 (1939). [MF 288]

Let $F(x)$ be a monotonic function which is singular (that

is, continuous, not constant, and having $F'(x)=0$ almost everywhere); and let

$$c_n(F)=(2\pi)^{-1}\int_{-\pi}^{\pi}e^{-inx}dF(x)$$

be its Fourier-Stieltjes coefficients. As $|n|\to\infty$, it is possible to have $c_n(F)=o(1)$ [Menchoff: see Zygmund, Trigonometrical Series, 1935, 294–295]; $c_n(F)=o(|n|^{-\alpha})$ for some $\alpha>0$ [Littlewood, Quart. J. Math., Oxford Ser. **7**, 219–226 (1936)]; $c_n(F)=o(|n|^{-1+\epsilon})$ for any $\epsilon>0$ [Wiener and Wintner, Amer. J. Math. **60**, 513–522 (1938); J. Math. Phys. Mass. Inst. Tech. **17**, 233–246 (1939)]. The author shows that, for any positive function $r(n)$ such that $r(n)\uparrow\infty$ as $n\uparrow\infty$, it is possible to have $c_n(F)=O\{r(|n|)|n|^{-1}\}$. The method is that of Wiener and Wintner [in the first paper cited], considerably simplified by the use of an idea of F. Riesz [Math. Z. **2**, 312–315 (1918)], which the author applies in obtaining an auxiliary function.

R. P. Boas, Jr. (Durham, N. C.).

Cameron, R. H. and Wiener, Norbert. Convergence properties of analytic functions of Fourier-Stieltjes transforms. Trans. Amer. Math. Soc. **46**, 97–109 (1939). [MF 14]

This is an extension of the investigations of Wiener and Pitt [Duke Math. J. **4**, 420–435 (1938)] on absolutely convergent Fourier-Stieltjes integrals, where they showed that the reciprocal of such an integral is itself an integral of the same type under certain circumstances. Instead of the reciprocal, the present authors consider general multiple-valued analytic functions, and, instead of absolute convergence, that is, finiteness of $V_{-\infty}^{\infty}f(y)$, they admit other norms depending upon a parameter θ, $0<\theta\leqq 1$, which reduce to the total variation when $\theta=1$. One of these norms is

$$T_\theta\{F(x)\}=2\sum_{-\infty}^{\infty}\left[\int_n^{n+1}df(y)\right]^\theta,\quad F(x)=\int_{-\infty}^{\infty}e^{-ixy}df(y).$$

Both norms have the properties $T(F_1+F_2)\leqq T(F_1)+T(F_2)$, $T(F_1F_2)\leqq T(F_1)\cdot T(F_2)$, $T_\theta(aF)=|a|^\theta T_\theta(F)$. Let $h(x)$ be the discrete and $s(x)$ the singular part of $f(x)$, and let $H(x)$ and $S(x)$ be the corresponding FS-transforms. Let $T_\theta\{F(x)\}<\infty$, let R be the closure of the set of values of $F(x)$, and let R^* be the set of complex numbers whose distance from R is not greater than $\{T_\theta[S(x)]\}^{1/\theta}$. Let $R_0\supset R^*$ be an open set and $\mathfrak{F}(z)$ an n-valued ($n\geqq 1$) analytic function having no singularities in R_0. Form the $2n$ continuous functions $\mathfrak{F}(F(x))$ and $\mathfrak{F}(H(x))$. To every branch $[\mathfrak{F}(F(x))]_j$ we can order two branches $[\mathfrak{F}(H(x))]_{p_j}$ and $[\mathfrak{F}(H(x))]_{q_j}$ (p_j and q_j uniquely determined) such that the quadratic mean square difference of $[\mathfrak{F}(F(x))]_j$ and $[\mathfrak{F}(H(x))]_{p_j}$ on the interval $(0,+\infty)$ is zero as well as that of $[\mathfrak{F}(F(x))]_j$ and $[\mathfrak{F}(H(x))]_{q_j}$ on $(-\infty,0)$. If for any particular j, $p_j=q_j$, then $T_\theta\{[\mathfrak{F}(F(x))]_j\}<\infty$. The complicated proof is based upon showing that a function is of finite norm in this metric if and only if it is locally of finite norm in a certain sense. *E. Hille.*

Cramér, Harald. On the representation of a function by certain Fourier integrals. Trans. Amer. Math. Soc. **46**, 191–201 (1939). [MF 125]

The author discusses the representation of a bounded measurable complex-valued function $f(t)$ in the form

$$(1)\qquad \int_{-\infty}^{\infty}e^{itx}d\alpha(x)$$

for various classes of functions $\alpha(x)$. Let $\mu(t)$ have the properties

$$\mu(0)=1;\quad \int_{-\infty}^{\infty}|\mu(t)|dt<\infty;\quad \mu(t)=\int_{-\infty}^{\infty}e^{ixt}m(x)dx,\quad m(x)\geqq 0$$

(for example, $\mu(t)=e^{-\beta|t|}$ or $\mu(t)=(1-|t|)$ for $|t|\leqq 1$, $\mu(t)=0$ for $|t|>1$). Define

$$g_\epsilon(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixt}\mu(\epsilon t)f(t)dt,\qquad \epsilon>0.$$

Corresponding respectively to the cases (a) $\alpha(x)$ bounded and non-decreasing, (b) $\alpha(x)$ on $(-\infty,\infty)$ of bounded variation, (c) $\alpha(x)$ the integral of a function $g(x)\epsilon L(-\infty,\infty)$, the author obtains the following necessary and sufficient conditions for $f(x)$ to have the representation (1):

(a) $$g_\epsilon(x)\geqq 0,\quad 0<\epsilon<1;\quad -\infty<x<\infty;$$

(b) $$\int_{-\infty}^{\infty}|g_\epsilon(x)|dx<\text{const.},\qquad 0<\epsilon<1;$$

(c) $g_\epsilon(x)$ satisfies (b) and

$$\lim_{\epsilon,\epsilon'\to 0}\int_{-\infty}^{\infty}|g_\epsilon(x)-g_{\epsilon'}(x)|dx=0.$$

If $f(x)$ has one of the representations in question, the appropriate condition is satisfied for all $\mu(t)$ having the properties stated above; if $f(x)$ satisfies the condition for just one $\mu(t)$, it has the appropriate representation. In cases (b) and (c) the boundedness of $f(x)$ may be replaced by integrability on every finite interval. All the conditions can be generalized in a straightforward way to functions of k real variables ($k=1,2,\cdots$). In particular, condition (a) with a suitably specialized $\mu(t)$ is a simplification of the criterion of S. Bochner [Math. Ann. **108**, 407 (1933)] for characteristic functions of k-dimensional distribution functions.

R. P. Boas (Durham, N. C.).

Boas, R. P., Jr. On a generalization of the Stieltjes moment problem. Trans. Amer. Math. Soc. **46**, 142–150 (1939). [MF 16]

The author discusses the generalized moment-problem

$$(I)\qquad \mu_n=\int_0^{\infty}t^{\lambda_n}d\alpha(t),\quad 0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty,$$

with regard to the uniqueness of the unknown function $\alpha(t)$, non-decreasing in $(0,\infty)$, when we say (I) is determined.

The study of (I) is made by discussing the growth of

$$f(z)=\int_0^{\infty}t^z d\alpha(t),$$

which is analytic for $R(z)>0$. Various sufficient conditions for the determinate character of (I) are given. To illustrate, (I) is determined if $|\lambda_n-n|<A$ ($n=1,2,\cdots$) and

$$\lim_{n\to\infty}\lambda_n^{-1}\mu_n^{1/2n}=0.$$

J. A. Shohat (Philadelphia, Pa.).

Friedrichs, Kurt and Horvay, Gabriel. The finite Stieltjes momentum problem. Proc. Nat. Acad. Sci. U. S. A. **25**, 528–534 (1939). [MF 268]

Let $L_0=1, L_1, L_2, \cdots, L_r$ be $(n+1)$ first moments which are given. The authors investigate conditions which must be satisfied by L_i, $i=0,\cdots,n$, and by n numbers $\lambda_1<\lambda_2<\cdots<\lambda_n$ in order that the part of the spectrum

of a solution of the corresponding moment problem, which does not exceed λ_n, would reduce precisely to the set $(\lambda_1, \lambda_2, \cdots, \lambda_n)$. They find the following conditions: Let $P_m(x)=(x-x_1)\cdots(x-x_m)$, $P_0(x)=1$, and $L_{m+j,\,m}\equiv L_{m+j,\,m}(x_1, \cdots, x_m)$ be the result obtained from $x^jP_m(x)$ if the powers of x are replaced by the corresponding moments. Let ${}^r\Omega_m(x_1, x_2, \cdots, x_m)$ be the determinant of order $(r+1)$, whose ith row is $L_{m+i-1,\,m}, L_{m+i,\,m}, \cdots, L_{m+i-1+r,\,m}$ $(i=1, 2, \cdots, r+1)$. Then the conditions in question are (*) ${}^r\Omega_m(\lambda_1, \lambda_2, \cdots, \lambda_m)>0$, $0\leqq r$; $0\leqq m$; $m+2r\leqq n$, where the inequality ${}^0\Omega_n>0$ must be replaced by the equality ${}^0\Omega_n=0$ if $\lambda_1, \lambda_2, \cdots, \lambda_n$ exhaust the whole spectrum.

The proof is based on the remark that the result obtained from the polynomial $(\sum_{i=0}^{r}\gamma_ix^i)^2(x-\lambda_1)\cdots(x-\lambda_m)$ by replacing the powers of x by the corresponding moments, is necessarily positive; here $\gamma_0, \cdots, \gamma_r$ are arbitrary real constants. The authors point out that conditions (*) are not sufficient. Various properties of the functions ${}^r\Omega_m$ are discussed and the nature of regions in the space of variables $(\lambda_1, \cdots, \lambda_n)$ determined by inequalities (*) is investigated. The obtained results appear to be useful in the problem of approximate calculation of characteristic values of various boundary value problems. *J. D. Tamarkin.*

Wald, Abraham. Limits of a distribution function determined by absolute moments and inequalities satisfied by absolute moments. Trans. Amer. Math. Soc. **46**, 280–306 (1939). [MF 127]

The paper is devoted to the solution of the following two problems. 1. Given the absolute moments $M_\nu=E|X-x_0|^\nu$ $(\nu=1, 2, \cdots, j)$ of the chance variable X about the given point x_0. Find the least upper and the greatest lower bounds of the probabilities $P[-d<Y-x_0<d]$, for all chance variables Y, with $E|Y-x_0|^\nu=E|X-x_0|^\nu$ $(\nu=1, 2, \cdots, j)$. This is a generalization of results due to Tchebycheff and Markoff $(j=1)$, to Guldberg, Cantelli and others $(j=2)$, and to Tchebycheff, Markoff, Stieltjes $(i_\nu=\nu;\ a_\nu=E(X-x_0)^\nu;$ $\nu=1, 2, \cdots, j)$. 2. Find necessary and sufficient conditions that the given positive numbers a_ν $(\nu=1, 2, \cdots, j)$ shall be the absolute moments of a chance variable X about the given point x_0, of given order $i_1, i_2, \cdots, i_j$, respectively. The solution is achieved by means of "arithmetic chance variables" (X is such if $\sum_{i=1}^{k}P(X=x_i)=1$, for a finite system of distinct values $x_1, x_2, \cdots, x_k$). The author studies convergent sequences of such arithmetic chance variables, also the number of changes of sign of functions of the type $D(X)=P(X<x)-P(Y<x)$ (the notions of convergence and change of sign having been properly defined). The general results thus obtained are illustrated for $j=2, 3$. Thus, if M_r, M_s, M_t are given $(r<s<t)$, then the following conditions are necessary and sufficient for the solution of Problem 1: $M_r\geqq 0$, $M_s\geqq M_r^{s/r}$, $M_t\geqq M_r(M_s/M_r)^{(t-r)/(s-r)}$.

J. A. Shohat (Philadelphia, Pa.).

* **Szegö, Gabor. Orthogonal Polynomials.** American Mathematical Society Colloquium Publications, v. **23**. American Mathematical Society, New York, 1939. ix+401 pp. $6.00.

This is the first detailed systematic treatment of orthogonal polynomials (a Fascicule in the "Mémorial des Sciences Mathématiques," devoted to the same topic, is of necessity limited in content). As stated in the Preface, no claim is made for completeness. Thus, very little space is devoted to the problem of moments and to the relation of orthogonal polynomials to continued fractions. However, these omissions are compensated by several interesting features: (a) an elaborate treatment of the asymptotic behaviour of orthogonal polynomials, by various methods, with applications, in particular, to the "classical" polynomials of Legendre, Jacobi, Laguerre and Hermite; (b) a detailed study of expansions in series of orthogonal polynomials, regarding convergence and summability; (c) a detailed study of orthogonal polynomials in the complex domain; (d) a study of the zeros of orthogonal polynomials, particularly of the classical ones, based upon an extension of Sturm's theorem for differential equations. The book presents many new results; many results already known are presented in generalized or more precise form, with new simplified proofs.

The book consists of 16 chapters as follows. Chapter I is devoted to necessary preliminaries: Weierstrass' Approximation Theorem, bounds for trigonometric polynomials, linear functional operations, Gamma, Bessel and Airy Functions, "Comparison Theorems" of Sturm's type. Chapters II, III treat the existence of orthogonal polynomials, their various representations and general properties, closure, recurrence relation, nature and general distribution of zeros. We find here also special examples, the classical polynomials and their generalizations and polynomials corresponding to certain discrete distributions. An interesting feature is the discussion of the celebrated Tchebycheff Inequalities, for which 3 proofs are given. The relation to continued fractions is briefly sketched. Chapters IV–IX are devoted almost exclusively to the classical polynomials and their direct generalizations, including the related functions of the second kind. A detailed study is made, by various methods, of their zeros, also of their asymptotic behaviour, for real and complex values of the variable, if we let the degree of the polynomial in question increase indefinitely. The asymptotic formulae derived employ not only the customary trigonometric functions $\cos nx$, $\sin nx$, but also Bessel Functions, a considerable improvement. They are applied to a study of expansion problems: convergence, equi-convergence, summability. We find here also (Chapter VII) bounds for other special classes of orthogonal polynomials (for example, case of monotonic weight functions). Chapter X deals with representation of positive functions, which serves as an important tool in the study of orthogonal polynomials on the unit-circle (Chapter XI) and of their important relations to polynomials orthogonal on a real interval. This study is continued in Chapter XII (asymptotic properties) and in Chapter XIII (expansion problems); it is extended to a rectifiable Jordan curve in Chapter XVI. Chapters XIV, XV deal with Lagrangean and Hermitian interpolation and with mechanical quadratures, with or without a weight-function, mainly for the classical polynomials.

The book closes with a list of interesting problems and exercises and a considerable bibliography.

J. A. Shohat (Philadelphia, Pa.).

Heuser, Paul. Zur Theorie der Faberschen Polynomreihen. Deutsche Math. **4**, 451–454 (1939). [MF 131]

The original definition of the polynomials of Faber is based on the assumption that the given curve C is an analytical one; a corresponding expansion theorem holds provided the given function $F(z)$ is analytic in the closed interior of C. The author deals with the case in which C is a rectifiable curve (or even slightly more general) and $F(z)$ is regular in the interior of C and continuous on C. By ap-

proximating C from the exterior, he derives a Faber expansion for $F(z)$ which is Abel-summable in the interior of C. Various details of the paper remained unclear to the reviewer. *G. Szegö* (Stanford University, Calif.).

Schoblik, Friedrich. Über eine Funktionalbeziehung Hermitescher Polynome. Monatsh. Math. Phys. **47**, 333–337 (1939). [MF 64]

For the Hermite polynomials

$$H_n(x)=(-1)^n e^{x^2}(e^{-x^2})^{(n)},$$

the product $H_n(x+y)H_n(x-y)$ can be represented as a linear combination of

$$H_{2\nu}(2^{1/2}x)H_{2n-2\nu}(2^{1/2}y),\quad \nu=0, 1, 2, \cdots, n.$$

A similar but more complicated formula for the two-dimensional analogues of the Hermite polynomials is also obtained. *G. Szegö* (Stanford University, Calif.).

Jackson, Dunham. A class of orthogonal functions on plane curves. Ann. of Math. **40**, 521–532 (1939). [MF 91]

An order relation for the monomials $y^q\cos px$, $y^q\sin px$ is defined and the Schmidt process of orthogonalization on a curve C is applied. This process does not involve any difficulty if we exclude the case that a certain monomial vanishes identically on C. On the other hand, in the latter case certain modifications of the process are necessary which are discussed in the first part of the paper. Then the derivation of an analogue to the recurrence formula and to the Christoffel-Darboux identity is indicated, and elementary expansion theorems are discussed. The second part deals with analogous considerations for the monomials $\cos px\cos qy$, $\cos px\sin qy$, $\sin px\sin qy$. *G. Szegö.*

Sheffer, I. M. Some properties of polynomial sets of type zero. Duke Math. J. **5**, 590–622 (1939). [MF 176]

A set of polynomials $\{P_n\}$ $(n=0, 1, 2, \cdots)$ is said to be of zero type if there is an operator J

$$J[y]\equiv c_1y'+c_2y''+c_3y'''+\cdots,$$

so that $J[P_n]=P_{n-1}$. Appell sets $(J[y]\equiv y')$ and difference polynomial sets $(J[y]=y(x+1)-y(x))$ are special sets of zero type.

The author discusses properties of sets of type zero and obtains generating functions of such sets and several functional equations satisfied by polynomials of such sets. To every set there belongs a unique operator J, but to every operator infinitely many sets $\{P_n\}$. One and only one of these sets, which is called the basic set, has the property $B_0(0)=1$, $B_n(0)=0$ $(n=1, 2, 3, \cdots)$. Besides the above definition of zero type sets, some equivalent characterizations are given. If $\{P_n\}$ is of type zero and K an operator of form

$$K[y]\equiv k_1y'+k_2y''+k_3y'''+\cdots,$$

then the sets $\{K[P_{n+1}]\}$, $\{K^2[P_{n+2}]\}$, $\cdots$ are also of zero type. Examples of sets of type zero are $P_n\equiv x^n/n!$, Laguerre, Hermite, and Newton polynomials.

Special consideration is given to zero type sets possessing a functional equation which is of finite order when written with a suitable operator K. Necessary and sufficient conditions for $\{P_n\}$ being a set of this class, and properties of such sets, are investigated. All zero type sets which are orthogonal are determined. Some of the results are generalized to sets of higher type. *A. Erdélyi* (Edinburgh).

Differential Equations

Rádl, Franz. Über die Teilbarkeitsbedingungen bei den gewöhnlichen Differentialpolynomen. Math. Z. **45**, 429–446 (1939). [MF 49]

The author's generalized division process for linear differential polynomials [Math. Z. **31**, 451–456 (1929); **40**, 375–386 (1935)] is examined with a view towards describing the structure of a pair of polynomials of which one is divisible by the other in the generalized sense. The methods permit the complete discussion of pairs of polynomials of low orders. *J. F. Ritt* (New York, N. Y.).

Taussky, Olga. An algebraic property of Laplace's differential equation. Quart. J. Math., Oxford Ser. **10**, 99–103 (1939). [MF 160]

The Dirac equations form a generalization to four dimensions of the Cauchy-Riemann equations. The author investigates similar generalizations to any number of dimensions n and arrives, among others, at the following results: Let $l_1, \cdots, l_n$ be n linear combinations with constant coefficients of the first partial derivatives of the functions $u_1, \cdots, u_n$. In order that n similar first order linear functional operators applied to the l's should reduce to the Laplacians of $u_1, \cdots, u_n$ respectively, it is necessary that n must be such that there exists a hypercomplex system, not necessarily associative, with n base elements, with respect to the real numbers, without divisors of zero. It is shown that n must reduce to 1, 2, 4, 8, 16, or to a multiple of 16. *H. Poritsky* (Schenectady, N. Y.).

Langer, Rudolph E. The boundary problem of an ordinary linear differential system in the complex domain. Trans. Amer. Math. Soc. **45**, 151–190 and **Correction**, 467 (1939). [MF 124]

In this interesting paper on linear boundary problems, Langer extends to two-dimensional regions X of the complex x-plane various general results hitherto obtained only for an interval of the real x-axis; the methods used are the natural modifications of those based on asymptotic properties and contour integration, which have been so successfully employed in the real case; but the theory is so presented as to be applicable to any simply-connected continuum of which an arbitrary pair of points can be joined by an arc in X of uniformly bounded length. No special cases are given to illustrate the scope of the expansion in two-dimensional regions.

The familiar boundary problem refers to an ordinary linear differential system, written

$$Y'(x,\lambda)=\{\lambda\mathfrak{R}(x)+\mathfrak{Q}(x)\}\mathfrak{Y}(x,\lambda)$$

in matrix notation; here, if we turn to the two-dimensional case, $\mathfrak{R}(x)$ and $\mathfrak{Q}(x)$ may be taken as bounded and analytic, while λ, as usual, is a complex parameter. It is also assumed that the n roots $r_i(x)$ of the determinantal equation in ρ, $|\mathfrak{R}-\rho\mathfrak{I}|=0$ are everywhere distinct, with $|r_i(x)-r_j(x)|>d>0$. To begin with, Langer prepares such a system so that $\mathfrak{R}(x)$ takes the normal form $(\delta_{ij}r_i(x))$, while $q_{ii}(x)=0$ for $i=1, 2, \cdots, n$. There is a well-known formal solution

$$\mathfrak{P}(x,\lambda)(\delta_{ij}e^{\lambda R_i(x)}),\qquad R_i'=r_i,$$

where $\mathfrak{P}(x,\lambda)$ is a matrix of formal power series in $1/\lambda$ of which the initial term reduces to $\mathfrak{I}$; if, for any λ in the chosen λ region, there exist $n(n-1)$ fixed points $x_*^{(i,j)}$ of X (actually on its boundary) such that any point x of X may be joined to each $x_*^{(i,j)}$ by an arc of uniformly bounded length along which the corresponding $\lambda[R_i(x)-R_j(x)]$ has

a non-increasing real-part, then Langer shows (Part I) that there exists an actual solution $\mathfrak{Y}(x, \lambda)$, analytic in x and λ, and asymptotically represented by the formal matrix for $|\lambda|$ large, at least to terms of the kth degree in $1/\lambda$, k being an arbitrary integer not less than 0; this representation will be valid in X and a certain sector of the complex λ-plane, forming "associated regions" by definition. Here the points $x_\sigma^{(i,j)}$ replace the two endpoints of the interval serving as limits of integration in the real case. Langer treats from the outset the more general case when $\mathfrak{Q}(x)$ is replaced by $\mathfrak{Q}(x, \lambda)$, analytic in x and λ, and asymptotically represented by a matrix series in powers of $1/\lambda$.

The familiar boundary problem refers also to a set of linear boundary conditions involving in the real case the two endpoints, or more generally, any set of m points $\eta_1, \eta_2, \cdots, \eta_m$:

$$\sum_{\rho=1}^{m} \mathfrak{W}^{(\rho)}\mathfrak{Y}(\eta_\rho, \lambda)=0$$

[see C. E. Wilder, Trans. Amer. Math. Soc. **18**, 416 ff. (1917)]. Here $\mathfrak{Y}(x, \lambda)$ is a matrix solution of the given differential system, having identical columns. Langer shows (Part II) that if x is a "fundamental region" (that is, associated with a finite number of λ regions covering the λ region in question) then known methods suffice to determine the asymptotic distribution in the λ-region of the characteristic values of λ for which such a vector solution exists. Langer treats indeed a more general problem in which $\mathfrak{W}^{(\rho)}$ are analytic in λ and represented by asymptotic series in $1/\lambda$. He also studies the adjoint boundary problem and the Green's matrices, and gives a novel form for the latter in the real case.

In Part III Langer passes to the consideration of the corresponding expansion of an arbitrary analytic vector in a linear series of these characteristic functions. Here he essentially assumes $\mathfrak{Q}(x, \lambda)$ to be independent of λ, and $\mathfrak{W}^{(\rho)}$ to be polynomial matrices in λ, and that the whole plane is a fundamental region. The Green's matrices are then analytic in the finite λ-plane except for poles, and the familiar contour integral formula applies for the summation of such a series as a set of residues. By a direct asymptotic study he shows that the series will converge to the assigned vector at a point x_0 of X if the boundary conditions are "regular," relative to x_0; and if these are regular for all x_0, the series will converge uniformly. The conditions for "regularity" are much more complicated than in the classical case of a real interval and two-point boundary conditions, but this complication seems inherent in the problem itself.

The form which the relation of biorthogonality takes in the event of multiple characteristic values is also discussed in Part III under the condition that the "index" is equal to the "multiplicity." *G. D. Birkhoff* (Cambridge, Mass.).

Rosenblatt, Alfred. Sur les points singuliers des équations différentielles. C. R. Acad. Sci. Paris **209**, 10–11 (1939). [MF 216]

This note states that its author has studied the system of differential equations

$$(\sum A y_1^{\alpha_1} y_2^{\alpha_2} x^{\beta}) \frac{dy_1}{dx} = \sum A^1 y_1^{\alpha_1^1} y_2^{\alpha_2^1} x^{\beta^1},$$

$$(\sum A y_1^{\alpha_1} y_2^{\alpha_2} x^{\beta}) \frac{dy_2}{dx} = \sum A^2 y_1^{\alpha_1^2} y_2^{\alpha_2^2} x^{\beta^2}$$

with particular reference to solutions of the form

$$y_1 = v_1 x^{\rho_1} e^{-k_1(\log 1/x)^m}, \quad y_2 = v_2 x^{\rho_2} e^{-k_2(\log 1/x)^m}, \quad 0<m<1.$$

A number of equations and conditions satisfied by v_1, v_2, k_1, k_2, and m are listed. It is stated that a method of successive approximations is used in the proofs. *W. M. Whyburn.*

Germay, R. H. J. Généralisation de l'équation de Hesse. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 139–144 (1939). [MF 108]

Let $Y_1, Y_2, \cdots, Y_{n+2}$ be linear functions of the variables $y_1, y_2, \cdots, y_n$ with coefficients dependent on the variable x. The author solves the partial differential equation

$$\frac{\partial u}{\partial x}+(Y_1-y_1Y_{n+2})\frac{\partial u}{\partial y_1}+\cdots+(Y_n-y_nY_{n+2})\frac{\partial u}{\partial y_n} = Y_{n+1}-uY_{n+2}$$

by reduction to a system of $n+2$ linear ordinary differential equations. *E. Rothe* (Oskaloosa, Iowa).

Bădescu, Radu. On a problem of Goursat. Gaz. Mat. **44**, 571–577 (1939). (Rumanian) [MF 73]

Treating a special case of a problem of Goursat in the theory of second order partial differential equations of hyperbolic type [Goursat, Ann. Fac. Sci. Univ. Toulouse, 2nd sér. **4** (1904)], the author determines explicitly the solution $z(x, y)$ of the equation $\partial^2 z/\partial x \partial y = z$, which reduces to preassigned analytic functions along the lines $x=x_0$, $y=y_0$ in a certain neighborhood of the point (x_0, y_0). *I. J. Schoenberg* (Waterville, Me.).

Sundaram, S. Minakshi. On non-linear partial differential equations of the parabolic type. Proc. Indian Acad. Sci., Sect. A. **9**, 479–494 (1939). [MF 1]

Using the methods of Lichtenstein and Siddiqi, the author studies the equation

$$(1) \qquad \frac{\partial}{\partial x}\left[\rho(x)\frac{\partial u}{\partial x}\right]-\frac{\partial u}{\partial t}=f\left(x, t, u, \frac{\partial u}{\partial x}\right)$$

for four types of boundary conditions of which a sample is (a) $u(x, 0)=u_0(x)$, (b) $u(0, t)=u(\pi, t)=0$; $t \geqq 0$. The method consists of looking for a solution of the form

$$(2) \qquad u(x, t)=\sum u_n(t)\phi_n(x),$$

where $\phi_n(x)$ are eigenfunctions for the associated boundary value problem

$$\frac{d}{dx}\left[\rho\frac{d\phi}{dx}\right]+\lambda\phi=0; \quad \phi(0)=\phi(\pi)=0.$$

Assuming the right member f can be expanded in a series of eigenfunctions $\phi_n(x)$, replacing u in equations (1) by the series (2) the author gets a system of non-linear integral equations for the functions $u_n(t)$ which is treated by the method of successive approximations. The principal difference between this and previous papers [for example, C. R. Acad. Sci. Paris **203**, 1120 (1936)] is that in place of being a power series in u and $\partial u/\partial x$, f here satisfies a Lipschitz condition. *E. W. Titt* (Hyattsville, Md.).

Sundaram, S. Minakshi. On non-linear partial differential equations of the hyperbolic type. Proc. Indian Acad. Sci., Sect. A. **9**, 495–503 (1939). [MF 2]

By the method of a previous paper (see the above review) the author treats the equation

$$\frac{\partial}{\partial x}\left[\rho(x)\frac{\partial u}{\partial x}\right]-\frac{\partial^2 u}{\partial t^2}=f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right)$$

for the boundary conditions

$$u(0, t)=u(\pi, t)=0; \quad u(x, 0)=u_0(x); \quad u_t(x, 0)=u_1(x).$$

E. W. Titt (Hyattsville, Md.).

Green, John W. Harmonic functions in domains with multiple boundary points. Amer. J. Math. **61**, 609–632 (1939). [MF 19]

This paper contains results related to material in papers by Perkins: The Dirichlet problem for domains with multiple boundary points [Trans. Amer. Math. Soc. **38**, 106–144 (1935)] and by Maria and Martin: Representation of positive harmonic functions [Duke Math. J. **2**, 517–529 (1936)].

The author shows that for a suitably restricted finite domain T of euclidean space the sequence solution of the extended Dirichlet problem (as defined by Perkins) can be written in the form

$$u(M)=\int_t u(\gamma)dm(\epsilon_\gamma, M),$$

where M is an arbitrary interior point of T, $u(\gamma)$ is a continuous function of an element γ of the boundary t of T, and $m(\epsilon, M)$ a mass distribution on the elements of t. He also develops an extension of a formula given by Maria and Martin [loc. cit.]. A function $u(P)$ positive and harmonic in a suitably restricted T may be represented in the form

$$u(P)=\int_t f(\gamma, P)d\mu(\epsilon_\gamma),$$

where $f(\gamma, P)$ is a function of P and of the element γ of t, but is independent of u, and $\mu(\epsilon)$ is a mass distribution on the elements of t. *F. W. Perkins* (Hanover, N. H.).

Horn, J. Über eine hypergeometrische Funktion zweier Veränderlichen. Monatsh. Math. Phys. **47**, 359–379 (1939). [MF 67]

The pair of differential equations

$$\text{(A)} \qquad xr+ys+(c-x)p=bz, \quad xs+yt+(c-y)q=b'z$$

is studied in the neighborhood of the singular point $x=\infty$, $y=0$ by first seeking a formal solution of type

$$z=\sum C_{m,n}x^{\rho-m}y^{\sigma+n}$$

as in Borngasser's dissertation. The indicial equations can be satisfied by either $\rho=-b$, $\sigma=0$ or by $\rho=-b$, $\sigma=b-c+1$. The resulting power series in x^{-1} are divergent but their coefficients are power series in y which are fully convergent. A third formal solution is obtained by putting $z=e^xZ$ and

$$Z=\sum C_{m,n}x^{\rho-m}y^n.$$

The equations (A) are next replaced by a set of 5 partial differential equations of the first order in which z, p, q are the dependent variables and the quantity $(x-y)^{-1}$, which appears as a coefficient, is expanded in ascending powers of y. Solutions are then sought of type

$$z=\sum z_nx^{-b-n}, \quad p=\sum p_nx^{-b-n}, \quad q=\sum q_nx^{-b-n},$$

and also of type

$$z=e^x\sum z_nx^{b-c-n}, \quad p=e^x\sum p_nx^{b-c-n}, \quad q=e^x\sum q_nx^{b-c-n}.$$

It is found that z_n, p_n, q_n are linear homogeneous functions of z_0, q_0 and polynomials in y; z_n and q_n are of degree n and p_n of degree $n-1$, while $p_0=0$. The quantity z_0 is found to be a solution of the differential equation

$$yz''(y)+(c-b-y)z'(y)-b'z(y)=0,$$

which has the two solutions $F(b; c-b; y)$, $y^{b-c+1}F(b+b'+1; b-c+2; y)$. The second type of solution leads to a series

$$z=e^xx^{b-c}\sum f_n(y)x^{-n}.$$

When y is kept constant and is restricted to a certain region in the complex plane or to a certain range of real values, the differential equations may be satisfied by Laplacian integrals of type $z=\int_0^\infty e^{-x\zeta}w(\zeta)d\zeta$, and these integrals may be represented asymptotically by the series as $x\to+\infty$ through positive values, the representation being uniform if y belongs to a certain region.

In Math. Ann. **113**, 269–272 (1936), the behavior of the solutions of the equations (A) as $x\to\infty$ is studied not only for the case in which y is kept constant, but also for the case in which the singularity $x=\infty$, $y=0$ is approached along the hyperbola $x=at$, $y=1/kt$. These two cases are now comprised in a more general investigation. A theorem is first proved for a system of differential equations with a parameter and the equations (A) are then replaced by a system of homogeneous integral equations in which the variable occurs as one limit of all the integrals.

H. Bateman (Pasadena, Calif.).

Integral Equations and Functional Analysis

Trjitzinsky, W. J. General theory of singular integral equations with real kernels. Trans. Amer. Math. Soc. **46**, 202–279 (1939). [MF 126]

This paper generalizes the theories of T. Carleman [Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala, 1923] to some more comprehensive classes of symmetric kernels. While Carleman's kernels ("rank 1") are limits of kernels, for which Hilbert's theorems about the continuous case are still true or, roughly speaking, for which $\int\int K(x, y)^2dxdy$ exists ("rank 0"), Trjitzinsky investigates some kernels of "rank 2," which are in some sense limits of kernels of rank 1, and further some analogously defined kernels of arbitrary finite rank. He supposes that the singularities of these $K(x, y)$ can be projected in such a way in some reducible closed set of points E on the x-axis, and that he may get a kernel $K_0(x, y)$ of rank 0 changing into 0 the values of K in the interior of a suitably constructed set of intervals Δ. Then, following the formation of Δ, K will become an iterated limit of kernels, the first of which is K_0. According to the number of steps he needs, he defines K to be in one class of a sequence of classes H_n, the first of them H_1 containing Carleman's kernels; he shows by an example that there are kernels in H_2 which do not belong to H_1. To develop his theory he needs the existence of a certain specified linear operator associated with $K(x, y)$, which is the exact generalization of a linear operator used by Carleman. The author shows that there exist kernels in H_2 with such associated operators, and the same could be proved for $H_3, \cdots$; these are the kernels he is investigating. Following the pattern of Carleman's theory he uses the iterated limit process mentioned above and he investigates what will carry over from the theorems and formulas of the spectra theory of Hilbert (class H_0) and Carleman (class H_1) by this process. Completing his proof by mathematical induction he succeeds in extending the most important results of Carleman to his kernels of H_n. Finally he goes on to define classes with "transfinite index"; he gets them supposing that the first vanishing derivative of the reducible set E, mentioned above, has a transfinite order. The author gives an example

of a kernel which is in H_{n+1} but not in a class with a lower index; he shows how to treat such kernels.

E. D. Hellinger (Evanston, Ill.).

Rothe, Erich. Topological proofs of uniqueness theorems in the theory of differential and integral equations. Bull. Amer. Math. Soc. **45**, 606–613 (1939). [MF 59]

Let $f(r)=r+F(r)$ be a transformation in a Banach space with $F(r)$ completely continuous. For a sphere V with surface S let $j(f, r)$ be the index of f at a point r in V [Leray-Schauder, Ann. École Norm. **51**, 46–78 (1934)] and let $u(f, s, \eta)$ be the order of f at η [Rothe, Compositio Math. **5**, 177–197 (1937)]. The author shows that, if f is locally univalent or has a nonsingular continuous differential along a continuous curve $r=r(t)$ in V, then $j(f, r)$ is invariant along the curve. It follows that the equation $f(r)=\eta$ has exactly one solution if $u(f, s, \eta)=\pm 1$, and any two solutions of $f(r)=\eta$ can be connected by a curve as described above. This uniqueness theorem is applied to the solution of a system of non-linear integral equations,

$$u_i(s)+\int f_i(s, t, u_1, \cdots, u_n)dt=0, \quad i=1, \cdots, n$$

for which an existence theorem had previously been obtained [Rothe, loc. cit.]. In formula (1.12) and also in the following formula, $u(l(r-r_0, r_0), s, f(r_0))$ should probably be replaced by $u(l(r-r_0, r_0), s, 0)$ or by $u(l(r-r_0, r_0)+f(r_0), s, f(r_0))$.

I. Halperin (Kingston, Ont.).

Nakano, Hidegorô. Über Abelsche Ringe von Projektionsoperatoren. Proc. Phys.-Math. Soc. Japan **21**, 357–375 (1939). [MF 207]

To the operations of logical sum, product and limit of point sets, we have corresponding operations in an Abelian set of projections. If the set is closed under these operations it is termed an Abelian Ring of Projections. The relation between this notion and normal operators, in particular with respect to the resolution of the latter in the complex plane, is discussed. *F. J. Murray* (New York, N. Y.).

Kantorovitch, L. The method of successive approximations for functional equations. Acta Math. **71**, 63–97 (1939). [MF 211]

The basis is a linear space Y on which there is defined a norm $|y|$, having the usual properties, with values in the positive part of a semi-ordered space Z, in which every finite set is bounded, and every bounded set has a least upper and greatest lower bound, combining the Banach and semi-ordered spaces. Such a space is called a B_δ space. By the method of iteration starting from the element $z=0$, there is proved the existence of a solution of the equation $V(z)=z$, where V is on $0\leqq z\leqq z'$ to Z, monotonic nondecreasing, continuous via monotone nondecreasing sequences, and such that $V(0)\geqq 0$ and $V(z')\leqq z'$. This leads to an existence theorem for solutions of $U(y)=y$, where U is dominated by such a V in the sense that $|U(y+\Delta y)-U(y)|\leqq V(z+\Delta z)-V(z)$, and $|U(0)|\leqq V(0)$, where $|y|\leqq z'$, $|\Delta y|\leqq \Delta z$ and $|y+\Delta y|\leqq z'$. Application is made to the case where f is additive and dominated by an additive continuous positive function $g(z)$ ($g(z)\geqq 0$, for $z\geqq 0$), in the sense $|f(y)|\leqq g(|y|)$. Special cases of the theorem mentioned include mostly well-known theorems for equations in infinitely many variables, integral and differential equations. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Kober, Hermann. Transformationen von algebraischem Typ. Ann. of Math. **40**, 549–559 (1939). [MF 94]

A linear Hilbert space operator T is of algebraic type if $T(Tf)$ is defined whenever Tf is defined and, for some polynomial $P(x)$ with simple roots, $\{P(T)\}f=0$ for all such f. It is characterized by the following: For some finite number of linearly independent linear subspaces $\mathfrak{M}_q$, $q=1, 2, \cdots, m$, and different constants l_q, $q=1, 2, \cdots, m$, Tf is defined if and only if f has the form $\psi_1+\psi_2+\cdots+\psi_m$, $\psi_q \varepsilon \mathfrak{M}_q$, and then $Tf=l_1\psi_1+l_2\psi_2+\cdots+l_m\psi_m$. The necessary and sufficient condition that T be bounded, involving only the $\mathfrak{M}_q$, is that for some constant A, $\|\psi_q\|\leqq A\|\psi_1+\psi_2+\cdots+\psi_m\|$ for all $\psi_1, \psi_2, \cdots, \psi_m$ with $\psi_q \varepsilon \mathfrak{M}_q$. Fourier, Hankel and other transforms are given as illustrations. *I. Halperin* (Kingston, Ont.).

Oxtoby, J. C. and Ulam, S. M. On the existence of a measure invariant under a transformation. Ann. of Math. **40**, 560–566 (1939). [MF 95]

The object of this paper is to prove, by the standard methods of this theory, the following theorems:

Theorem 1. Let T be an automorphism of a complete separable metric space E. In order that there exist a finite invariant measure in E it is necessary and sufficient that for some point p and compact set $C \subset E$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_C(T^k p)>0,$$

where f_C is the characteristic function of C. It is sufficient that the limit superior be positive.

Theorem 2. In order that an automorphism T of a complete separable metric space E admit a finite invariant measure which is zero for points, it is necessary and sufficient that one of the following conditions hold: (1) T has non-denumerably many periodic points; (2) there is a compact set C consisting of non-periodic points some one of which returns to C with positive frequency (or superior frequency) under iteration of T.

Theorem 3. A precise analogue of Theorem 1 for a suitably defined "steady flow" produced by a one-parameter group T_λ.

Theorem 4. Let T be an automorphism of a metric space E, complete, separable, and dense in itself. There exists in E a finite or infinite measure $m(A)$, which is invariant under T, defined for Borel sets, zero for points, positive (possibly infinite) for non-void open sets, and such that there is a set A_0 of finite measure whose images T^kA_0, $k=-\infty, \cdots, +\infty$, are all disjoint and together exhaust all but a finite amount of the measure in E.

B. O. Koopman (New York, N. Y.).

Dunford, Nelson. A mean ergodic theorem. Duke Math. J. **5**, 635–646 (1939). [MF 179]

The mean ergodic theorem of von Neumann is generalized to continuous n-parameter groups and semi-groups of linear transformations in a general Banach space. For groups the result is: Let T_α, $\alpha \varepsilon E_n$, be a group of linear transformations in a Banach space B. Assume that the transformations T_α are uniformly bounded, that $\bar{x}T_\alpha x$ is Lebesgue measurable for every x in B and $\bar{x}$ in the conjugate space, and that to each x in B corresponds a zero set N such that the points $T_\alpha x$, $\alpha \varepsilon E_n - N$, form a separable subset of B. Then for any family of cubes I_r with edge r, the

$$\lim_{r\to\infty} \frac{1}{r^n} \int_{I_r} T_\alpha x d\alpha$$

will exist for all x in a certain closed linear manifold $M \subset B$. The manifold M coincides with B in case B is reflexive, has

a weakly compact unit sphere, or equivalent uniformly convex norm. The method of proof is an extension of one used by F. Riesz [J. London Math. Soc. 13, 274–278 (1938)] and K. Yosida [Proc. Imp. Acad., Tokyo 14, 292–294 (1938)]. The result is applied to the group of transformations induced in the space L_q, over a measure space E, by a group of point transformations of E. *J. C. Oxtoby.*

Calculus of Variations

Graves, Lawrence M. The Weierstrass condition for multiple integral variation problems. Duke Math. J. 5, 656–660 (1939). [MF 181]

Let I be the multiple integral $\int_R f(t, x, p)dt$, where $t=(t_1, \cdots, t_n)$, $x=(x^1, \cdots, x^m)$, and p is the matrix $(p_\alpha^i)=(\partial x^i/\partial t_\alpha)$. The E-function is

$$E(t, x, p, P)=f(t, x, P)-f(t, x, p)-(P_\alpha^i-p_\alpha^i)f_i^\alpha(t, x, p),$$

where $f_\alpha^i=\partial f/\partial p_\alpha^i$. If $x=x(t)$ minimizes I in the class of D' manifolds with given boundary values, it is known that E is nonnegative whenever (t, x, p) is on the minimizing manifold and $P-p$ has rank 1. The author gives a new and very simple proof of this result. He then proves the analogous theorem for parametric integrals $I=\int g(x, j)dt$, the integrand being a function of the x and of the n-square minors of (p_α^i). This has previously been established only for $n=1$ and $n=m-1$. Let ρ be a symbol denoting a subset consisting of n of the integers $1, \cdots, m$. A matrix (p_α^i) determines a vector j_ρ whose components are the n-square minors of (p_α^i). A vector j_ρ determines by certain prescribed rules a matrix k with m rows and $m!/[(m-n+1)!(n-1)!]$ columns. In order that j_ρ represent the minors of some (p_α^i), it is necessary and sufficient that k have rank $m-n$. For a minimizing manifold the condition

$$E(x, j, J)=g(x, J)-g(x, j)-(J_\rho-j_\rho)g_\rho(x, j)\geqq 0$$

must hold for all (x, j) on the minimizing manifold and all J such that the corresponding matrix K has rank $m-n$ and (k, K) has rank not greater than $m-n+1$.

E. J. McShane (Charlottesville, Va.).

Reid, William T. Isoperimetric problems of Bolza in non-parametric form. Duke Math. J. 5, 675–691 (1939). [MF 183]

The principal contribution of this paper consists in the proof of an extension of the Lindeberg theorem to a problem of Bolza in non-parametric form in which some of the side-conditions are isoperimetric conditions. This is accomplished by adding to the integrand function $f(x, y, y')$ a term $l(x)\varphi_\alpha\varphi_\alpha$, where $\varphi_\alpha(x, y, y')=0$ are the differential side-conditions of the problem, and then determining $l(x)$ in such a way that the problem associated with the new integrand satisfies stronger forms of the conditions of Clebsch and Weierstrass. By means of this Lindeberg theorem the author obtains readily an important sufficiency theorem for the problem under consideration. Furthermore he establishes an Osgood theorem for such problems and he simplifies some of the proofs of an earlier paper [Ann. of Math. 38, 662 (1937)]. *A. Dresden* (Swarthmore, Pa.).

Douglas, Jesse. Green's function and the problem of Plateau. Amer. J. Math. 61, 545–589 (1939). [MF 17]

This paper presents a solution of the Plateau problem in the following form. Given an aggregate Γ of k non-intersecting Jordan curves in n-space, to find a minimal surface bounded by Γ and having a prescribed genus h and a prescribed orientability character (one-sided or two-sided). The surface is represented by a harmonic vector $H(u, v)$, where (u, v) varies over a Riemann surface R; for one-sided surfaces R is supposed symmetric and H is supposed to have the same value at symmetrically located points. Each representation $g(z)$ of Γ on the boundary of R determines a harmonic function $H(u, v)$; the Dirichlet integral of H is denoted by $A(g, R)$. By a familiar argument based on semi-continuity and compactness, it is shown that under suitable hypotheses concerning degenerate manifolds the functional $A(g, R)$ assumes its minimum. The proof that the minimizing harmonic function H satisfies $E=G$, $F=0$ is the central feature of this paper. This is effected by the construction of a particular variation of the Riemann surface R for which the variation of the Green's function of R assumes a simple explicit form, from which the conditions $E=G$, $F=0$ appear as ready consequences of the vanishing of the first variation of $A(g, R)$. The theorem finally established is the special case for finite h and k of the more general theorem stated in the next review. *E. J. McShane* (Charlottesville, Va.).

Douglas, Jesse. The most general form of the problem of Plateau. Amer. J. Math. 61, 590–608 (1939). [MF 18]

The methods outlined in the preceding review can be extended with little change to problems in which the genus and number of boundaries may be infinite. The following problem is studied. Given a Riemann surface (or semi-surface) R with boundary C, and given in n-space a topological image Γ of C, to prove the existence of a minimal surface topologically equivalent to R and bounded by Γ. The functional $A(g, R)$ is defined as in the preceding review. The class of Riemann manifolds topologically equivalent to R is denoted by T, while T' is the class of manifolds which are limits of T but do not belong to T. The numbers $d(\Gamma, T)$ and $d(\Gamma, T')$ are the greatest lower bounds of $A(g, R)$ for all R in the respective classes T, T' and all topological mappings $g(z)$ of C on Γ. Also, $e(\Gamma, T)=d(\Gamma, T')-d(\Gamma, T)$. The principal theorems are as follows. If $d(\Gamma, T)<d(\Gamma, T')<\infty$, there exists a minimal surface of the type of R bounded by Γ; this has least area for its type and boundaries. If there is a sequence $\Gamma_n\to\Gamma$ and $T_n\to T$ for which $\limsup e(\Gamma_n, T_n)>0$, there exists a minimal surface of the type of R bounded by Γ; each interior portion of this surface has an area which is finite and is the least possible for its own type and boundaries. *E. J. McShane.*

Dynamics, Celestial Mechanics, Relativity

Vincent, J. J. The mathematical theory of shuttle projection. J. Textile Inst. 30, T103–T126 (1939). [MF 165]

The author studies the exact motion of a loom shuttle when the elasticity of the picking mechanism is taken into account, but its mass is neglected. Under various mathematical assumptions he arrives at differential equations of the type $\ddot{x}+\lambda x=F(t)$. Seven different cases of possible motions are discussed. *W. Feller* (Providence, R. I.).

Johnsen, Leif. Calcul symbolique des pseudocoordonnées. Arch. Math. og Naturvid. 42, no. 3 (1939). [MF 129]

A symbolic differential calculus of pseudo-coordinates, introduced by the author in a previous paper [Avh. Norske Vid. Akad. Oslo. I. Mat.-Naturv. Kl., No. 11 (1937)], is here elaborated. Using these methods a new kinematical interpretation of "non-holonomic deviation" is obtained for the linear case. *D. C. Lewis* (Durham, N. H.).

Chazy, Jean. Sur une loi corrective de la loi de Newton. C. R. Acad. Sci. Paris **209**, 133–136 (1939). [MF 221]

A modified law of gravitational attraction

$$F=-f\frac{mm'}{r^2}\left(1+\epsilon\frac{dr}{dt}\right)$$

was considered by Popovici [Bull. Astr. **3**, 257–261 (1923); C. R. Acad. Sci. Paris **208**, 2052 (1939)] and by Armellini [Atti Accad. Naz. Lincei **26**, 209–215 (1937); **27**, 609–614 (1938)], primarily to provide a basis for a cosmogonic theory.

The suggested form of F is one of the possible mathematical interpretations of an assumption of a finite velocity V of propagation of gravitational attraction, ϵ being inversely proportional to V. It is shown that the observed motions of the planets are compatible with this modified law only if ϵ corresponds to a velocity V many times as great as the velocity of light, in agreement with results obtained by Laplace and others with different but similar assumptions for F.

It is remarked that the only well-established deviation from Newtonian motion of the principal planets, namely, the advance of the perihelion of Mercury, can be accounted for by a modification of the Newtonian law proportional to the square of v/c, the ratio between the planetary velocity and the velocity of light, and that the proposed law introduces a corrective term of the first order in v/V.

D. Brouwer (New Haven, Conn.).

Fabre, Hervé. Sur les solutions périodiques du problème des perturbations. C. R. Acad. Sci. Paris **209**, 291–293 (1939). [MF 233]

The problem of perturbations of planets and of asteroids of the solar system, which we consider with each planet perturbed separately (the planetary problem of three bodies), presents itself in two different aspects according as there exists, or does not exist, a simple approximate commensurability of the mean motions that can affect the eccentricity, the inclination, and other elements of an orbit. The author discusses a case not treated by Poincaré and others, considering first and second order terms, a case in which an inclination of the orbit is not zero. He finds, for feeble eccentricities and inclinations, that there is a zone of unstable motions. *E. J. Moulton* (Evanston, Ill.).

Moghe, D. N. On the stability of motion in Milne's kinematical system. Proc. Indian Acad. Sci., Sect. A. **10**, 41–44 (1939). [MF 79]

It is found that, although Milne has taken a static space to describe the motion of a system of particles according to his Kinematical Theory of Relativity, this motion is not in general stable; it is found to be stable only in the particular case given by $G(\zeta)=-1$, which determines the so-called substratum in its essence. *Extract from the paper.*

McVittie, G. C. The cosmical constant and the structure of the universe. Observatory **62**, 192–194 (1939). [MF 76]

The author re-determines the radius of the closed Friedmann universe on following Einstein's suggestion of allowing the cosmological constant to vanish, and finds that this would require a high density ($\sim 10^{-27}$ y/cm^3) of matter.

H. P. Robertson (Princeton, N. J.).

Mechanics of Continua

Mohr, Ernst. Die laminare Strömung längs der Platte und damit verwandte Flüssigkeitsbewegungen. Deutsche Math. **4**, 477–513 (1939). [MF 134]

Roy, Maurice. Sur l'écoulement relatif permanent d'un fluide parfait et l'hypothèse des tranches. C. R. Acad. Sci. Paris **209**, 276–279 (1939). [MF 230]

In a previous note [C. R. Acad. Sci. Paris **209**, 187 (1939)] the author has indicated a transformation of the Euler equations of a perfect fluid containing the entropy of the fluid. In the present note he draws further conclusions from the form of these equations. *E. Reissner.*

Høiland, Einar. On the interpretation and application of the circulation theorems of V. Bjerknes. Arch. Math. og Naturvid. **42**, no. 5, 69 pp. (1939). [MF 130]

This paper is based on the circulation theorems for a perfect fluid due to V. Bjerknes, the more useful being $\int a_T dm=\int g_T dm$; the integrals are taken round a thin tube of constant cross-section; a_T, g_T are tangential components of acceleration and body force, respectively; dm is the element of mass. Using this theorem the author claims to establish necessary and sufficient conditions for stability in the case of a fluid at rest or in rotation as a vortex under the action of gravity. In general the fluid is either homogeneous or heterogeneous and incompressible. Flow between parallel planes is also discussed. The arguments are not mathematically convincing. *J. L. Synge* (Toronto, Ont.).

Müller, Wilhelm. Zum Problem der Längsbewegung eines Flugzeugs. Z. Angew. Math. Mech. **19**, 193–202 (1939). [MF 139]

The equations of motion are written in the form

$$\dot{v}=-g\sin\phi-k_v v^2,\quad v\dot{\phi}=k_a v^2-g\cos\phi,$$

where ϕ is the angle which the tangent to the path makes with the x-axis. The product $v\cdot k$ is put equal to $f(t)$ and the integration is first performed for the case in which $f(t)$ is a constant. The more interesting case in which $f(t)=c/(t+t_0)$ leads to an error integral with a complex argument. The case in which the computations can be made with Fresnel integrals is discussed. Some indications are given of an application to looping motions.

H. Bateman (Pasadena, Calif.).

Kampé de Fériet, J. Les fonctions aléatoires stationnaires et la théorie statistique de la turbulence homogène. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 145–210 (1939). [MF 109]

This paper represents an application of the ideas of E. Slutsky and A. Khintchine to the paper by G. I. Taylor: Diffusion by continuous movements [Proc. London Math. Soc. **20**, 196–211 (1921)]. Taylor considers the time means of the motion of a particle which at each of a succession of instants has a certain probability of conserving its velocity, and a certain probability of reversing its velocity. The author considers means in similar situations, not with respect to the time, but with respect to a parameter of randomness. As the author states in a footnote, the scope of the paper is closely related to that of the reviewer: Generalized harmonic analysis [Acta Math. **55**, 117–258 (1930)], but is brought into more direct contact with the work of the Russian school on random functions. The harmonic analysis of random functions is studied. The paper belongs to the kinematics of turbulence rather than to its dynamics, as the equations of hydrodynamics are not introduced.

N. Wiener (Cambridge, Mass.).

Savin, S. A. Flexure of a rectangular covering slab elastically fixed by its four sides. J. Math. Phys. Mass. Inst. Tech. **18**, 239–255 (1939). [MF 194]

Uniform compressive loads p and p_0 act on the faces $z=c$ and $z=-c$, respectively, of a rectangular slab. The behavior of the slab is to be described under these loads, under its own weight and under certain initially undetermined forces and moments acting on the remaining faces $x=\pm a$, $y=\pm b$. Using a method described by the author in a previous paper [J. Math. Phys. Mass. Inst. Tech. **17**, 201 (1939)], polynomial expressions for the components u, v and w of the displacement are written down as solutions of the differential equations of equilibrium. From these expressions for the components of the displacement are calculated the normal and the shearing stresses on the faces $x=\pm a$, $y=\pm b$, and the couples on these faces. The prescribed conditions on the faces $z=c$ and $z=-c$ are shown to be satisfied. *H. W. March* (Madison, Wis.).

Willers, Fr. A. Die Beullast abgestufter Kreisplatten. Z. Angew. Math. Mech. **19**, 206–210 (1939). [MF 140]

The buckling load is determined for circular plates with piecewise constant thickness; the influence of various distributions is discussed. *K. O. Friedrichs.*

Hay, G. E. The method of images applied to the problem of torsion. Proc. London Math. Soc. **45**, 382–397 (1939). [MF 251]

The author presents a method of solving the St. Venant torsion problem for the regions bounded by (a) a rectangle, (b) an equilateral triangle, (c) an isosceles right triangle, and (d) a 30° right triangle. The solution of the equation

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=-2, \tag{1}$$

subject to the condition $\phi=0$ on the boundary of the given region, is obtained by superimposing on the particular integral of (1) (in the form of the logarithmic potential due to a uniform surface distribution over the region in question) a doubly infinite set of logarithmic potentials produced by the surface distributions over a polygonal network of regions external to the given region. The method is illustrated by a formal calculation of the torsion function and the torsional rigidity of the 30° right triangle.

I. S. Sokolnikoff (Madison, Wis.).

Theory of Probability

Greenwood, Joseph A. and Greville, T. N. E. On the probability of attaining a given standard deviation ratio in an infinite series of trials. Ann. Math. Statistics **10**, 297–298 (1939). [MF 155]

Let x_n denote mutually independent stochastic variables, each of which takes on the values 0 or 1 with the probability $q=1-p$ and p, respectively. The authors propose to prove that for any $k>0$ the probability that $(x_n-np)/\sqrt{npq}\leqq k$ for some n is equal to 1. This is, however, a simple special case of familiar results and far better limitations for the probability in question are well known (Khintchine's law of the iterated logarithm). *W. Feller* (Providence, R. I.).

Kac, M., van Kampen, E. R. and Wintner, Aurel. On Buffon's problem and its generalizations. Amer. J. Math. **61**, 672–676 (1939). [MF 24]

The well known problem in geometrical probability of determining the "probable" number of intersections of a system of equally spaced parallel lines and a plane convex curve C, when C is thrown at random on the plane of the parallel lines, is considered. The authors show that the classical choice of the "probability measure" is the only one which satisfies the assumptions that are tacitly made in the usual treatment of this problem. After "probability" has been defined, a simple calculation yields a formula for the Fourier-Stieltjes transform of the distribution of the number of intersections in terms of the supporting function of C. The classical results may easily be deduced from this formula. *P. Hartman* (New York, N. Y.).

Dantzig, George B. On a class of distributions that approach the normal distribution function. Ann. Math. Statistics **10**, 247–253 (1939). [MF 149]

The author introduces a class of sequences defined by a simple recurrence formula and which approach the normal distribution in the same sense as the binomial coefficients do. As an application the author studies the distribution of the number of inversions in the group of all permutations of n elements as $n\to\infty$. *M. Kac* (Ithaca, N. Y.).

Marcinkiewicz, Józef. Quelques théorèmes de la théorie des probabilités. Bull. Sém. Math. Univ. Wilno **2**, 22–34 (1939). [MF 114]

The paper deals primarily with strong limit processes, involving probabilities of satisfying not merely a single inequality, but simultaneously a set of inequalities, finite or infinite.

The paper consists of three parts: (1) the solution of a problem proposed by P. Lévy; (2) some inequalities for sequences of independent chance variables; (3) applications.

As a solution of Lévy's problem, the author presents Theorem 2. Let x_n be independent chance variables, such that if $\xi>X$,

$$c\xi^{-\alpha}\leqq\text{Prob}\,\{|x_n|>\xi\}\leqq C\xi^{-\alpha},\qquad 0<\alpha<1.$$

Let $\lambda(t)$ be an increasing function such that the oscillation of $\log\lambda(t)$ between t and $2t$ becomes indefinitely small as $t\to\infty$. Make the additional condition that the expected value of each x_n be zero. Then the probability for an infinite number of realizations of

$$|S_\nu|>|\nu\log\nu\lambda(\log\nu)|^{1/\alpha},\quad\text{where } S_\nu=\sum_1^\nu x_i,$$

is zero if the series $\sum|p\lambda(p)|^{-1}$ converges. Otherwise, it is unity.

The second part may be exemplified by Theorem 4. Suppose that, for $x>0$, Prob $(x_\nu>x)=$ Prob $(x_\nu<-x)$. Set

$$S=\sum_1^n x_i,\quad S^*=\max_{1\leqq\nu\leqq n}\sum_1^\nu x_i.$$

Then

$$\text{Prob}\,(S^*\geqq x)\leqq 2\ \text{Prob}\,(S\geqq x).$$

The last part contains among others a new proof of the following theorem of Khintchine [Rec. Math. (Moscou), N. S. **2** (1937)]: Let $x_{n,\nu}$ $(n,\nu=1,2,\cdots)$ be independent chance variables such that $\max_\nu$ Prob $(|x_{n,\nu}|>\epsilon)\to 0$, $(\epsilon>0)$. Then the limit law of the sum $\sum_\nu x_{n,\nu}$ is infinitely divisible.

E. L. Dodd (Austin, Tex.).

Kac, M. and van Kampen, E. R. Circular equidistributions and statistical independence. Amer. J. Math. **61**, 677–682 (1939). [MF 25]

The authors prove that if $f(x)$, $g(x)$ are two measurable functions of x, $0\leqq x\leqq 1$, the first of which is real and the

second of which is integrable, such that

$$\int_0^1 g(x) \exp(iuf(x))dx = 0$$

for $-\infty < u < +\infty$, then $\int_E g(x)dx = 0$ for every set $E = E_\omega$ on the x-interval $[0, 1]$ defined by the condition $g(x) < \omega$. Using the method of Fourier-Stieltjes transforms, it is shown that this result implies that, if $\varphi(x)$, $\psi(y)$ are real measurable functions, then the angular function $\{\varphi(x)+\psi(y)\}$ is statistically independent of y on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ if and only if either (i) the angular distribution function of $\varphi(x)$ is the circular equidistribution, while the distribution of $\psi(y)$ is arbitrary; or (ii) for a certain integer k, $k \geqq 1$, the angular distribution of $\varphi(x)$ over the circle $0 \leqq \varphi < 1$ is periodic with a primitive $1/k$, while for a number α, $0 \leqq \alpha < 1/k$, the probability is 1 that $\psi(y)$ attains one of the values $l/k+\alpha$, $l = 0, 1, \cdots, k$. It is understood that the usual definitions of distribution function and statistical independence are suitably modified for the case of angular functions, that is, functions whose values are to be reduced mod 1. *P. Hartman* (New York, N. Y.).

Lévy, Paul. Mouvement brownien linéaire et mouvement brownien plan. C. R. Acad. Sci. Paris **209**, 140–142, and **Errata**, 387 (1939). [MF 222]

Cette note précise et complète les résultats de trois notes antérieures [**207**, 1152 et 1368; **208**, 318 et Errata, 776].

Extract from the paper.

Feller, Willy. Die Grundlagen der Volterraschen Theorie des Kampfes ums Dasein in wahrscheinlichkeitstheoretischer Behandlung. Acta Bioth. Ser. A. **5**, 11–40 (1939). [MF 186]

The author makes a careful study of the development of a population whose number at time t, $N(t)$, is treated as a random function. Thus if it is supposed that the probability that each individual in a time interval of length dt has probability λdt of producing a second individual, the exact value of the probability of having n individuals at time t is found. The usual deterministic treatment gives $N(t) = N(0)e^{\lambda t}$; the probability treatment gives Expectation $N(t) = N(0)e^{\lambda t}$. It is then assumed (corresponding to the assumptions underlying the Pearl-Verhulst differential equation in the deterministic treatment) that the size of the population exerts a restricting influence on the birth rate proportional to its size. In this case the probability treatment gives a smaller expectation for $N(t)$ than the value obtained from the Pearl-Verhulst equation. This is also shown to be true in more general cases. The discussion then goes over to one in which $N(t)$ is no longer regarded as changing by integral jumps, so that the stochastic process considered becomes a "purely continuous" one instead of a "purely discontinuous" one, and differential equations are set up for the determining probability functions, following Kolmogoroff [Math. Ann. **104**, 415–458]. These are solved in certain important special cases, and results obtained analogous to those for the purely discontinuous case.

J. L. Doob (Urbana, Ill.).

Theoretical Statistics

Hartley, H. O. Recent advances in mathematical statistics. J. Roy. Statist. Soc. **102**, 406–444 (1939). [MF 196]

This is a bibliography of mathematical statistics for the second half of 1937 and for 1938.

von Mises, R. An inequality for the moments of a discontinuous distribution. Skand. Aktuarietidskr. **1939**, 32–36 (1939). [MF 122]

Consider a random variable assuming a sequence $\{x_i\}$ of real values with probabilities $p(x_i)$, $\sum p(x_i) = 1$. Let $c = \text{g.l.b.}|x_i - x_k|$ if $i \neq k$, and denote by M_ν $(\nu = 1, 2, \cdots)$ the νth absolute moment of this distribution about its mean. The author proves the inequality (1) $cM_\nu \leqq 2M_{\nu+1}$ $(\nu \geqq 1)$, which holds as far as the intervening moments exist. The constant 2 appearing on the right-hand side of (1) can not be lowered. The proof is quite elementary. No applications are made. *I. J. Schoenberg* (Waterville, Me.).

Lukomski, J. On some properties of multidimensional distributions. Ann. Math. Statistics **10**, 236–246 (1939). [MF 148]

Among the system of random variables $x_1, x_2, \cdots, x_n$ there may exist one or more linear connections, or linear bonds. Let the number k of such bonds in a complete system of such bonds be called the decrement of the distribution of the system of variables and let $n-k$ be the rank of the distribution. Another proof is given of Frisch's theorem [R. Frisch, Nordisk Stat. Tidskrift **1**, 36–102 (1929)] that the rank of the distribution is equal to the rank of the matrix of second order moments of the variables. Given the matrix of second order moments, it is shown how a complete system of linear bonds may be written out. Next, if the variables $k_1, k_2, \cdots, k_n$ be subjected to a linear transformation, the decrement of the distribution of the new variables is found, and finally this result is applied in the case where the original variables were subject to an n-dimensional normal distribution. *C. C. Craig* (Ann Arbor, Mich.).

Ottestad, Per. On the use of the factorial moments in the study of discontinuous frequency distributions. Skand. Aktuarietidskr. **1939**, 22–31 (1939). [MF 121]

Girshick, M. A. On the sampling theory of roots of determinantal equations. Ann. Math. Statistics **10**, 203–224 (1939). [MF 146]

The first part of the paper deals with the joint moments of two quantities q and Z defined by the relations

$$q = r_1^2 r_2^2 \cdots r_s^2, \quad Z = (1-r_1^2)(1-r_2^2)\cdots(1-r_s^2),$$

where the canonical correlations $r_1, r_2, \cdots, r_s$ are roots of the equation

$$\begin{vmatrix} -\lambda a_{ij} & a_{ik} \\ a_{kj} & -\lambda a_{kl} \end{vmatrix} = 0, \qquad \begin{matrix} i,j = 1,2,\cdots,s; \\ k,l = s+1,\cdots,s+t; \end{matrix} \qquad s \leqq t,$$

a_{mn} $(m, n = 1, 2, \cdots, s+t)$ being the second order product moments in a sample of $N > s+t$ observations from a multivariate normal population of $s+t$ variates in which the variates corresponding to $m = 1, 2, \cdots, s$ are independent of those corresponding to $m = s+1, \cdots, s+t$ $(s \leqq t)$. The quantities q and Z have been previously discussed to some extent by Hotelling [Biometrika **28** (1936)] and Wilks [Biometrika **24** (1932)].

The second part of the paper is devoted to a rather limited study of the relation between the asymptotic distribution of the canonical correlations and that of the latent roots of the matrix of sample variances and covariances for the variates corresponding to $m = 1, 2, \cdots, s$.

In the third part the author derives by the classical method of differentials asymptotic standard errors of the latent roots of a matrix of sample variances and covariances and of the coefficients of Hotelling's principal components. In a short section at the end the sampling theory of several quantities related to q and Z is considered. *S. S. Wilks.*

Kendall, M. G. and Smith, B. Babington. The problem of m rankings. Ann. Math. Statistics **10**, 275–287 (1939). [MF 152]

The problem which is considered in this paper is the following: If n objects are ranked by the ordinal numbers $1, 2, 3, \cdots, n$ with respect to some characteristic by each of m individuals, do the m rankings show any evidence of "community of judgment" among the m rankers? As an index of similarity of rankings the authors propose a quantity W defined as $(12S)/(m^2(n^3-n))$ when S is the sum of squares of deviations of the sums of ranks associated with the individual objects about the mean $m(n+1)/2$; W is such that the larger the value of W, the greater the similarity of rankings. The first four moments of W are determined under the assumption that ranking is done "at random" by each ranker, that is, under the assumption that all permutations of the ranks are equally likely and the m rankings are independent in the probability sense. The exact distributions of S are determined and tabulated for the cases $n=3$, $m=2$ to 10; $n=4$, $m=2$ to 6; $n=5$, $m=3$. The adequacy of fitting the Z distribution to the W distribution and hence using the Z test as an approximate significance test for W is discussed. The conclusion is reached that the Z test is a satisfactory approximate test for $n>3$. The results of the paper are illustrated by two examples.

S. S. Wilks (Princeton, N. J.).

Rietz, H. L. On the distribution of the "student" ratio for small samples from certain non-normal populations. Ann. Math. Statistics **10**, 265–274 (1939). [MF 151]

A description of interesting sampling experiments showing that the familiar ratio $\bar{x}/s$ follows pretty closely the "Student's" distribution even in cases where the sampled population is distinctly non-normal, causing a significant regression of s on $\bar{x}$. *J. Neyman* (Berkeley, Calif.).

Stock, J. Stevens and Frankel, Lester R. The allocation of samplings among several strata. Ann. Math. Statistics **10**, 288–293 (1939). [MF 153]

A population is divided into a number of strata and n_i denotes the number of individuals to be randomly drawn from the ith stratum. The sum $\sum n_i=n$, extended over all strata, is considered as fixed. The authors determine the values of the n_i, by which the estimates of the means of an individual character for particular strata are all equally accurate. *J. Neyman* (Berkeley, Calif.).

Yates, F. Tests of significance of the differences between regression coefficients derived from two sets of correlated variates. Proc. Roy. Soc. Edinburgh **59**, 184–194 (1939). [MF 162]

The author considers the case where the expected values of the random variables u_i and v_i are respectively $\epsilon(u_i)=a_1x_i+a_2y_i+a_3z_i$ and $\epsilon(v_i)=b_1\xi_i+b_2\eta_i+b_3\zeta_i$ for $i=1, 2, \cdots, n$. The values of x_i, y_i, z_i, ξ_i, η_i, and ζ_i are supposed to be known, while those of the regression coefficients a_j and b_j are not, and A_j and B_j denote the best unbiased estimates of a_j and b_j. The author deduces the formulae for the estimate of variance of the difference A_j-B_j. The case considered differs from the one to which the Markoff Theorem is directly applicable in that the residuals $u_i-\epsilon(u_i)$ and $v_i-\epsilon(v_i)$ are not assumed to be mutually independent.

J. Neyman (Berkeley, Calif.).

Pearson, E. S. Note on the inverse and direct methods of estimation in R. D. Gordon's problem. Biometrika **31**, 181–186 (1939). [MF 260]

Applications of the Theory of Probability, Economics

Vajda, Stefan. Die Wahrscheinlichkeit einer bestimmten Auszahlungssumme. Skand. Aktuarietidskr. **1939**, 10–21 (1939). [MF 120]

While an ideal mortality table would provide data for the most probable benefit payment under given circumstances, it does not yield direct information concerning the probability of given deviations from normal, and so fails to afford means of estimating the probability that a given fund to provide future death benefits shall be definitely exhausted in a given number of years. The function $v(m, a_m)$ giving the probability that exactly a_m persons die in the mth year of insurance becomes, for the extended problem, a function of the number s of persons taking out policies, and of the values $a_1, a_2, \cdots, a_{m-1}$. The author obtains expressions for $f(n, a)$, the probability that the payment sum for the first n years is a, and associated functions, in terms of the expressions $v(m, a_m)$, using the method of characteristic functions, since the general theorem of Tschebischeff is too inexact for practical use in this connection.

A. A. Bennett (Providence, R. I.).

Steller, E. T. Critical remarks on some important points in actuarial science. Verzekerings-Arch. **20**, 27–44, 49–60 (1939). (Dutch) [MF 83]

A survey of the meaning and the manner of application of some frequently occurring averages. *W. Feller.*

Berger, Alfred. Der Verschiebungssatz der Versicherungsmathematik. Monatsh. Math. Phys. **47**, 419–425 (1939). [MF 70]

Verf. definiert als mittlere Prämienreserve einer allgemeinen gemischten Versicherung das quadratische Mittel der künftig noch zu erwartenden Gewinne und Verluste. Für diese Grösse gilt ein Verschiebungssatz analog demjenigen der Wahrscheinlichkeitsrechnung. Ebenso steht sie in einfachem Zusammenhange mit der gewöhnlichen Prämienreserve und den Hausdorff'schen Formeln. *W. Saxer.*

Kerrich, J. E. A method of determining the rate of interest involved in a given transaction. J. Inst. Actuar. **70**, 198–210 (1939). [MF 123]

Suppose that $f(x)=c$ has one root ρ within the interval $x_k<x<x_{k+1}$.

Let $P_1(x)$ denote the polynomial of lowest degree passing through $f(x_1), \cdots, f(x_k), f(x_{k+1}), \cdots, f(x_n)$ and $P_2(x)$ the polynomial through $f(x_2), \cdots, f(x_k), f(x_{k+1}), \cdots, f(x_{n+1})$, where the arguments $x_1, x_2, \cdots$ are written in their natural order and $2<k<n$. It is shown that, if $f^{(n)}(x)$ does not change sign within the interval $x_1<x<x_{n+1}$, then in the interval $x_k<x<x_{k+1}$ $P_1(x)=c$ has a root ρ_1 and $P_2(x)=c$ has a root ρ_2, and that ρ lies between ρ_1 and ρ_2.

P. Johansen (Copenhagen).

Dodd, Edward L. The length of the cycles which result from the graduation of chance elements. Ann. Math. Statistics **10**, 254–264 (1939). [MF 150]

The problem dealt with in this paper is connected with the investigations of E. Slutsky, V. Romanovsky and others concerning sinusoidal configurations produced by repeated summations of chance elements. The author is dealing here with the wavy configuration resulting from a single application of a specified graduation formula to a series of chance variables. The chance elements are assumed to be normally and independently distributed about zero with constant variance. The length of the cycle produced by the gradua-

tion formula is defined with reference to a certain property of the graduated values and is equal to the reciprocal of the probability of that property. Properties like maxima, minima (in a certain sense), up-crossings and down-crossings of the base line are considered and the corresponding cycle-lengths determined. *A. Wald* (New York, N. Y.).

Tintner, Gerhard. Elasticities of expenditure in the dynamic theory of demand. Econometrica **7**, 266–270 (1939). [MF 4]

The results given in this paper are adaptations from an earlier paper by the author [Econometrica **6**, 375 ff. (1938)] in which he considered the influence of income, prices and interest rates upon the quantities of a set of commodities demanded at different points in time. In the present paper, the author considers the influence of the same factors upon expenditures. The relationships derived are expressed in the form of elasticities, thus making them independent of units of measurement. *S. S. Wilks* (Princeton, N. J.).

Wald, A. A new formula for the index of cost of living. Econometrica **7**, 319–331 (1939). [MF 215]

I shall give a new formulation of the index formula developed in an earlier paper [Z. Nationalökonomie, Wien, 1937], and shall express it as a function of the income. Further I shall discuss some properties of this formula and the criteria for its applicability. *Extract from the paper.*

Mathematical Physics

Herzberger, M. Normal systems with two caustic lines. J. Opt. Soc. Amer. **29**, 392–394 (1939). [MF 188]

With the object in mind to replace in a subsequent paper the conception of Sturm's conoid by another, equally simple, but not contradicting optical laws, the author investigates the existence of normal systems with two caustic lines. He shows that, disregarding the degenerate case of two parallel straight caustic lines, there exist only two such systems, one having toric wave surfaces, the other having cylindrical wave surfaces. *P. Boeder* (Southbridge, Mass.).

Herzberger, M. Theory of image errors of the fifth order in rotationally symmetrical systems. I. J. Opt. Soc. Amer. **29**, 395–406 (1939). [MF 189]

The purpose of the paper is to provide the designer of optical systems with a new tool to be used, for instance, in the design of high aperture lenses where the theory of third order aberrations (Seidel) as well as the tracing of meridional rays proves inadequate.

The first part of the paper contains the calculation of the image error coefficients up to the fifth order in terms of the coefficients of Hamilton's characteristic function (the eiconal of Bruns) for a finite object distance, the grouping of the image errors according to their order of aperture and field, and the derivation of the relations existing between the object and image side errors considered separately and jointly.

In the second part, the image error coefficients are determined under the assumption that both the object and the image surfaces are curved. The effect of a curved diaphragm on the image errors is also calculated.

The third part is devoted to the investigation of the change of the image errors resulting from a shift of the diaphragm and of the object. *P. Boeder.*

Weber, Ernst. The electrostatic field produced by a point charge in the axis of a cylinder. J. Appl. Phys. **10**, 663–666 (1939). [MF 118]

The electrostatic field due to a point charge on the axis of a cylindrical surface, when the latter is maintained at zero potential, is expressed as a Fourier-Bessel integral. The field along the axis and in the plane of symmetry is evaluated by replacing the Bessel functions in the integral by their asymptotic approximations. *H. Poritsky.*

Molière, G. Berechnung verallgemeinerter Gitterpotentiale. Z. Kristallogr., Mineral. Petrogr. Abt. A. **101**, 383–388 (1939). [MF 128]

Conditionally convergent lattice sums of type

$$\sum_l{}' \pm r_l^{-n}, \qquad n=1, 2, 3,$$

are considered for simple types of ionic lattices, the index symbol $l=(l_1, l_2, l_3)$ relating to a lattice point independent of its charge. A study is made also of the generalized lattice potential

$$\phi_n(r)=\sum_l\sum_k e_k|r-r_k^l|^{-n}.$$

Use is made of Ewald's device in which the integration is broken up into two parts after r_l^{-n} has been expressed as an integral of Laplace's type. The generalizations of Ewald's formula involve the functions

$$G(x)=2\pi^{-\frac{1}{2}}\int_x^\infty e^{-t^2}dt, \quad D(x)=4\pi^{-\frac{1}{2}}\int_x^\infty e^{-t^2}t^2dt,$$

$$L(x)=\int_x^\infty e^{-t}dt/t.$$

H. Bateman (Pasadena, Calif.).

Petiau, Gérard. Sur l'équation d'ondes d'un corpuscule à deux états de masse susceptible de représenter le proton-neutron. C. R. Acad. Sci. Paris **209**, 194–197 (1939). [MF 228]

This is an attempt to build up a system of two linear wave equations of Dirac type to represent a compound particle having two different masses, as for example the double particle proton-neutron. The proper values of the spin are shown to be $\pm h/4\pi$, $\pm 3h/4\pi$. The system of equations suggested applies only in the absence of an external field. The notion of electric charge is not introduced, and consequently there is no asymmetry between the properties of the two masses of the particle. *M. S. Vallarta.*

Heinrich, Gerhard. Resonanzschwingungen eines Systems bei vektorieller Überlagerung der erregenden Impulse. Z. Angew. Math. Mech. **19**, 216–223 (1939). [MF 142]

If a rotating system of natural frequency ω is acted on by a periodic torque of frequency ω_0, a necessary condition for resonance is that $\omega=k\omega_0$, where k is an integer. This is shown to be sufficient unless certain additional relations, which are explicitly found, hold. The results are extended to damped oscillations. *P. Franklin* (Cambridge, Mass.).

Niessen, K. F. und de Vries, G. Über die Empfangsimpedanz einer Empfangsantenne. I. Strahlungswiderstand. Physica **6**, 601–616 (1939). [MF 197]

Niessen, K. F. und de Vries, G. Über die Empfangsimpedanz einer Empfangsantenne. II. Reaktanz und Abbildungen. Physica **6**, 617–627 (1939). [MF 197]

A formula for the distribution of current along a receiving antenna is first derived. The current is that due to an incident signal wave whose electric field is parallel to the antenna. The rate of emission of energy from the wire

due to that current is then used to obtain the formulas for the resistance and reactance of the antenna for that radiation. Thus the problem of determining the impedance of the receiver is transformed into one of finding the impedance of a transmitter. The results first derived from the electromagnetic theory are subjected to rather severe approximations in order to obtain formulas which can be used to compute and show graphically the variation of resistance and reactance with the ratio of antenna length to the wave length of the signal. *R. V. Churchill* (Ann Arbor, Mich.).

GEOMETRY

*Larkins, James T., Jr. **Descriptive Geometry.** Prentice-Hall, Inc., New York, 1939. viii+317 pp. $2.50.

*Millar, A. V. and Shiels, K. G. **Descriptive Geometry.** D. C. Heath and Company, Boston, 1939. x+192 pp. $2.25.

Goodstein, R. L. Rational triangles. Math. Gaz. **23**, 264–269 (1939). [MF 208]

A general formula giving such triangles was stated by Gauss; we propose to find a general formula giving prime solutions, that is, a formula giving integral sides and circumradius, a, b, c, R, which have highest common factor unity. *Extract from the paper.*

van de Laar, W. Rotation and translation. Mathematica, Zutphen. B. **8**, 61–70 (1939). (Dutch) [MF 88]

When two equilateral triangles ABC, $A'B'C'$ lie in parallel planes, the vertices being taken in the same sense, and M_a, M_b, M_c are the centers of AA', BB', CC' respectively, then M_a, M_b, M_c are also the vertices of an equilateral triangle. This theorem is proved with the aid of quaternions, and generalized to two regular tetrahedrons.

D. J. Struik (Cambridge, Mass.).

Mehmke, Rudolf. Zur Geometrie der Polygone, Polyeder und Polytope in linearen Räumen beliebiger Stufe bei euklidischer und nichteuklidischer Massbestimmung. Math. Z. **45**, 405–428 (1939). [MF 48]

In the first half of this paper, the author collects evidence in favor of the following theorem: In Euclidean space of d dimensions, let x' be the centroid of the orthogonal projections of any point x on a number of h-spaces $(0<h<d)$ which are permuted among themselves by an irreducible group of congruent transformations leaving a point m invariant; then $\overrightarrow{mx'}=(h/d)\,\overrightarrow{mx}$. He is chiefly concerned with the case when the h-spaces belong to a regular polytope; the theorem is then a natural extension of certain results of E. Schönhardt: Über die Summe der Projektionen eines Vektors [Deutsche Math. **2**, 446–451 (1937)]. The theorem is proved when the polytope is a regular polygon or polyhedron, a measure-polytope (hyper-cube), or (for $h=1$, $d-2$, or $d-1$) a regular simplex. More general theorems are considered, in which the point x is replaced by a bound vector (Stab) or a bound bivector (Blatt).

The second half of the paper begins with an elegant proof of the following theorem (again suggested by Schönhardt's results): Let a point x in the plane of any triangle be projected in the directions of the three medians on the corresponding sides, giving points x_1, x_2, x_3; then the centroid of the triangle $x_1x_2x_3$ lies mid-way between x and the centroid of the given triangle. This is generalized in three ways: masses are associated with the vertices so as to displace the centroid, the triangle is extended to a general simplex, and the point x is again replaced by a vector or a bivector.

The "nichteuklidische Massbestimmung," referred to in the title, is nowhere used. *H. S. M. Coxeter.*

Kruppa, Erwin. Die affine duale Ebene und eine ihr aufgeprägte Massbestimmung in einer Abbildung auf den R_4. Monatsh. Math. Phys. **47**, 338–355 (1939). [MF 65]

In this paper dual complex numbers are admitted as affine coördinates (u, v) in the plane, that is, $u=x_1+\epsilon x_3$, $v=x_2+\epsilon x_4$, with $\epsilon^2=0$. To each point $p(u, v)$ corresponds a point $P(x_1, x_2, x_3, x_4)$ of the real Euclidean four-dimensional space R_4. The author represents P by its projections $P'(x_1, x_2)$ and $P''(x_3, x_4)$; thus, using a familiar kind of descriptive geometry for R_4, he can carry out graphic constructions that deal with points p. The introduction of infinite dual points proves expedient. Their images in R_4 are infinite straight lines belonging to a parabolic "Strahlkongruenz." The method yields constructions for join and intersection, cross-ratio, the locus of points having real cross-ratios with three fixed points (von Staudt's Kette), to which corresponds a parabola in R_4, and for a two-dimensional analogue. Next, the square of the distance between the points (u, v), (u', v') is defined by $(u-u')(\bar{u}-\bar{u}')+(v-v')(\bar{v}-\bar{v}')$, the bars indicating conjugate numbers ($-\epsilon$ instead of ϵ). Obviously this implies the Euclidean metric in the field P'. Based on that Hermitean distance notion, at last circles and the paths of some rigid motions are investigated. A fairly similar study of the ordinary complex plane has been developed by the author [Akad. Wiss. Wien, S.-B. **146**, 99–114 (1937)]. *A. E. Mayer* (London).

Weyl, Hermann. Addition to my note "On unitary metrics in projective space." Ann. of Math. **40**, 634–635 (1939). [MF 102]

An elementary proof, suggested by Chevalley, that under a change of the projective coördinate system both the unitary point-plane and point-point distances, $\|x\xi\|$ and $[[xx']]$ in projective space, take on a factor varying between the same fixed limits $K=\Gamma/\gamma$ and $1/K$. The notation is that of a previous note by the author [Ann. of Math. **40**, 141 (1939)], in which the result was first established.

N. H. McCoy (Northampton, Mass.).

Haenzel, G. Die Polarentheorie des linearen Strahlenkomplexes und seiner Strahlenkongruenzen. J. Reine Angew. Math. **181**, 45–60 (1939). [MF 6]

In a first part the author resumes his investigations on the polarity of a linear complex [J. Reine Angew. Math. **174**, 226–236 (1936); cf. also ibid. **166**, 167–181 (1932), and related researches by St. Jolles, e.g. Math. Z. **33**, 733–790 (1931)]. Using line coordinates (with the spherical fundamental relation $\Omega\equiv p_1^2+p_2^2+p_3^2+p_4^2-p_5^2+p_6^2=0$) an arbitrary linear complex Γ can be given by $p_6=0$. Then $\sum_1^5\lambda_i p_i=0$ means a linear complex Γ_0 in involution with Γ and at the same time a linear congruence C_1^1 in $p_6=0$. Further, Γ_0 and C_1^1 constitute perspective linear varieties $(\lambda_1, \cdots, \lambda_5)$. In these spaces $(\Gamma_0)^4$ and $(C_1^1)^4$ there are polarities induced by Ω. Two congruences or complexes (λ_i), (λ_i') are said to be conjugate if $\lambda_1\lambda_1'+\lambda_2\lambda_2'+\lambda_3\lambda_3'+\lambda_4\lambda_4'-\lambda_5\lambda_5'=0$. In that case each of both congruences is invariant for the skew involution of the other, and conjugate complexes are in involution.

The second part deals with the congruences C_n^n contained in Γ of equal bundle and plane orders. The lines of C_n^n that meet a given straight line l meet its polar line l' with respect to Γ; they are generators of a ruled surface R^{2n} of the degree $2n$ having l and l' as n-fold directrices. The congruence C_n^n is a four-dimensional linear variety $(R^{2n})^4$ with all R^{2n}'s as elements. Now, R^{2n} is the common line set of C_n^n and a congruence C_1^1 in Γ; hence $(R^{2n})^4$ and $(C_1^1)^4$ are perspective. The polarity in $(C_1^1)^4$ produces a polarity in $(R^{2n})^4$.

That is developed, in a third part, by the example of a C_2^2 of the first species. We quote the following theorem, which is connected with the Kummer configuration: Each C_2^2 of Γ is invariant for five skew involutions, so that the lines of C_2^2 are interchanged by involution; the fixed C_1^1's of the skew involutions are vertices on a polar C_1^1-pentahedron in $(C_1^1)^4$. The paper then describes the R^{2n} for a C_3^3, namely, the lines of Γ meeting a cubic of the complex Γ, and concludes with remarks on the analogous transcendental congruence for a helix of Γ. *A. E. Mayer.*

Algebraic Geometry

Gentry, Frank C. Cremona involutions determined by a pencil of surfaces. Bull. Amer. Math. Soc. **45**, 614–622 (1939). [MF 60]

Involutorial Cremona transformations defined by a pencil of surfaces of order n with an $(n-2)$-fold line have been discussed by Carroll [Amer. J. Math. **54**, 707–717 (1932)]. The present paper discusses the modifications produced in the transformations when the residual base-curve of the pencil is composite. The order of the transformation is unaltered unless the residual base-curve contains m conics each meeting the multiple line twice, in which case it is reduced by m. The configuration of fundamental curves of the second kind may be considerably modified, but their total number is always the same. *J. A. Todd.*

Tagg, E. D. Surfaces which contain an irrational pencil of rational curves. J. London Math. Soc. **14**, 216–228 (1939). [MF 246]

Using algebraic methods only, the author proves the following theorem due to Enriques: A surface containing an irrational pencil of rational curves can be transformed birationally into a cone. He states that the proof given by Enriques [Math. Ann. **52**, 449–456 (1899)] is unsatisfactory chiefly because in it a theorem of M. Noether [Math. Ann. **3**, 161–227 (1870). On page 228, the page number of this reference is given incorrectly] proved only for linear systems of curves is assumed to hold for non-linear systems.

T. R. Hollcroft (Aurora, N. Y.).

Mac Lane, Saunders and Schilling, O. F. G. Zero-dimensional branches of rank one on algebraic varieties. Ann. of Math. **40**, 507–520 (1939). [MF 90]

Let L be a field of algebraic functions of n variables over a ground field K. The following problem is investigated: Find all possible value groups Γ of archimedean rank 1 which can be realized by zero-dimensional valuations of L/K. It is well known that the rational rank of Γ (that is, the number of rationally independent elements of Γ) cannot be greater than n, and if it is equal to n, then Γ is a value group for L/K if and only if it is the direct sum of n cyclic groups. Surprisingly enough, it is shown that every value group of rank less than n is realizable on L/K. For this existence theorem two proofs are given, one based on a rational inductive construction of valuations in polynomial rings, the other making use of generalized Puiseux expansions. In connection with the second proof the following question is proposed: Can the defining power series of the valuation (for a suitable set of generators of L) be assigned in such a fashion that they be of ordinal type ω? It is known that the answer is affirmative in the case $n=2$.

O. Zariski (Baltimore, Md.).

Zariski, Oscar. The reduction of the singularities of an algebraic surface. Ann. of Math. **40**, 639–689 (1939). [MF 104]

Let $\{\xi_1, \cdots, \xi_n\}=\{\xi\}$ be the coordinates of the general point of a surface F in the n-dimensional affine space over an algebraically closed field K of characteristic 0. All rational functions on F are given by the field $K(\{\xi\})=\Sigma$. In order to describe the projective model of Σ, F we consider the field $\Sigma'=\Sigma(\omega_0)$, ω_0 transcendental over Σ. We understand by $\omega_0, \omega_i=\omega_0\xi_i$ the coordinates of the general point of the projective model. The varieties are described by the rings $J=K[\{\xi\}]$, $J'=K[\{\omega\}]$, respectively. We term varieties normal if the associated rings J, J' are integrally closed. A point $\mathfrak{p}$ of J is called a simple point if there exist two algebraically independent elements $x, y \in J$, say ξ_1, ξ_2, such that the prime ideal $\mathfrak{p}$ is an isolated prime component of $J(\xi_1-\alpha_1, \xi_2-\alpha_2)$, $\xi_i \equiv \alpha_i \pmod{\mathfrak{p}}$. A normal variety has only a finite number of singular points.

(I) The problem solved in this paper is to find a model of Σ all of whose points are simple. The branches of F defined by infinitely near points are described arithmetically in terms of valuations B on Σ. To every B belongs a valuation ring $\mathfrak{B}$ with a divisorless prime ideal $\mathfrak{P}$. If $\mathfrak{B} \supset J$ then $\mathfrak{P} \cap J$ is called the center of B on J. The following lemma is crucial for the local uniformization of singularities: Given any 0-dimensional valuation B of Σ there exists a projective model of Σ on which the center of B is a simple point.

(II) The proof of this lemma makes use of quadratic and general Cremona transformations which are determined by the values of the generators of Σ in 3-space. In order to study the properties of a surface in the neighborhood of a point $\mathfrak{p}$ it suffices to consider points at finite distance on an affine model J. In order to investigate the local aspects of surfaces the author introduces the quotient ring $Q(\mathfrak{p})$ of a point $\mathfrak{p} \subset J$ at finite distance. If $\mathfrak{p}$ is a simple point then $Q(\mathfrak{p})=J^*$ is integrally closed. Assuming that the projection of J on the (x, y)-plane $R=K[x, y]$ is the origin, it follows that the 0-dimensional prime ideal $\mathfrak{p}^*$ of J^* is equal to $J^*(x, y)$. As a direct consequence the author finds $J=(R, \mathfrak{p}^m)$, $J^*=(R, \mathfrak{p}^{*m})$, m arbitrary positive integers; that is, x and y are local uniformizing parameters at $\mathfrak{p}$. The important fact that all unmixed 1-dimensional ideals of J^* are principal yields that J^* is the totality of all elements in Σ which are mapped into the ring of holomorphic functions $K\{x, y\}$.

Let J, J' be two birational models of Σ. We say that $\mathfrak{p} \subset J$ corresponds to $\mathfrak{p}' \subset J'$ if there exists a 0-dimensional B such that $\mathfrak{p}$ is the center of B on J and $\mathfrak{p}'$ the center on J'. $\mathfrak{p}$ is called a fundamental point of the correspondence if there exists a 1-dimensional B_1 whose center on J is $\mathfrak{p}$ while center on J' is a curve Γ. Using the join $\bar{J}=(J, J')$ and the theory of contracted and extended ideals the author proves the following theorems: The locus of points $\mathfrak{p}'$ which correspond to a given point $\mathfrak{p}$ of J is an algebraic variety of dimension not greater than 1. If $Q(\mathfrak{p})$ is integrally closed and if $\mathfrak{p}$ is not a fundamental point, then to $\mathfrak{p}$ there corresponds a unique point $\mathfrak{p}'$ on J'.

(III) The point $\mathfrak{p}$ which is to be resolved can be chosen to be fundamental for a suitable quadratic transformation from some $K[\omega_0, \cdots, \omega_n]$, $\omega_i/\omega_0=\xi_i$ to $K[\omega_0\omega_j, \{\omega_i\omega_j\}]$. A care-

ful study of the quadratic transformation yields that every $\hat{\mathfrak{p}} \neq \mathfrak{p}$ on F has a unique map $\hat{\mathfrak{p}}' \subset F'$. Moreover $Q(\mathfrak{p}) = Q(\hat{\mathfrak{p}}')$. Using the theory of valuation ideals the author proves $Q(\mathfrak{p}) \subset Q(\mathfrak{p}')$ for singular points, that is, quotient rings are controlling characters of singularities.

(IV) Even if $Q(\mathfrak{p}) = J^*$ is integrally closed it does not necessarily follow that $Q(\mathfrak{p}') = J'$ is integrally closed. Let J' be the integral closure of J'. The 0-dimensional prime ideal of J' can split into several 0-dimensional prime ideals $\mathfrak{p}'$ of J'. (Theory of conductor.) Let J_1^* be the quotient ring of J' with respect to some $\mathfrak{p}'$. We shall say that J_1^* is obtained from J^* by a quadratic transformation followed by integral closure. An infinite sequence of quotient rings $J^* = Q(\mathfrak{p})$, $J_1^* = Q(\mathfrak{p}_1)$, $J_2^* = Q(\mathfrak{p}_2)$, $\cdots$ is called a normal sequence if (i) each $Q(\mathfrak{p}_i)$ is the quotient ring of a point $\mathfrak{p}_i$ on some normal projective model F_i of Σ; (ii) $Q(\mathfrak{p}_{i+1})$ is obtained from $Q(\mathfrak{p}_i)$ by a quadratic transformation followed by integral closure. Let $\Omega = \lim J_i^*$. Then Ω is the valuation ring of a 0-dimensional valuation B. This lemma implies the local reduction theorem: If $\{Q(\mathfrak{p}_i)\}$ is a normal sequence of quotient rings, then for some integer m, all points $\mathfrak{p}_i$, $i \geqq m$, are simple points.

(V) In the proof the theory of valuation ideals is used to set up the various quadratic transformations which are necessary to approximate Ω by quotient rings of simple points. (See II.) Finally, let M be the set of all normal sequences $\{Q(\mathfrak{p}_i)\}_\alpha$, where $\mathfrak{p} \subset F$ is fixed. Let $\mathfrak{p}_{h(\alpha)}$ be the first simple point of $\{Q(\mathfrak{p}_i)\}_\alpha$. Then let $g(\mathfrak{p})$ be the upper bound of all $h(\alpha)$. It is proved that $g(\mathfrak{p}) < \infty$. Moreover, $\mathfrak{p}$ is simple if and only if $g(\mathfrak{p}) = 0$. By (IV) it follows that $g(\mathfrak{p}_1) < g(\mathfrak{p})$ for the transformation $J^* \to J_1^*$. Using this numerical character attached to the finite number of singular points, (I), and observing (III) and (V), the author obtains the arithmetical proof of the theorem on the resolution of singularities. *O. F. G. Schilling* (Chicago, Ill.).

Coble, Arthur B. Configurations defined by theta functions. Duke Math. J. **5**, 479–488 (1939). [MF 166]

A review of some salient points of the invariant theory of Cremona transformations as developed by the author in previous papers. With every regular Cremona transformation in S_k, having n (or less) F-points, there is associated a linear homogeneous transformation, with integral coefficients, on $n+1$ variables which can be identified with the order and the multiplicities, respectively, of a linear system of hypersurfaces at the n points. These transformations form a group $G_{n,k}$. If the coefficients of the transformation are reduced modulo an integer ν, a finite factor group $G_{n,k}(\nu)$ is defined. On the other hand, under integral linear transformations of the periods, the odd and even theta-functions of genus p undergo the permutations of a finite group $M(2)$. The decisive remark is to the effect that the groups $G_{n,k}(2)$ and $M(2)$ are abstractly isomorphic, and that this isomorphism in many cases reflects an actual algebraic connection between point configurations and theta-functions. In particular this is so in the cases $p = 2, 3, 4$ and in the hyperelliptic case, because of the existence of three-term relations between the corresponding theta functions. Such a relation is namely a necessary condition for the algebraic connection to exist. *O. Zariski.*

Differential Geometry

***Hlavatý, Václav. Differentialgeometrie der Kurven und Flächen und Tensorrechnung.** P. Noordhoff N. V., Groningen-Batavia, 1939. xi+569 pp.

This textbook of differential geometry of curves and surfaces in ordinary space is a translation from the Czech. It represents an attempt to present the classical material with the full use of vector and tensor methods, and as such has some similarity to the first volume of Duschek-Mayer's textbook of 1930. Hlavatý's book contains (generally speaking) more material, and also differs in presentation, method and notation from its predecessor in this field.

The book is divided into four chapters. The first chapter, of 84 pages, is devoted to plane and space curves. It contains the customary topics, including natural equations and minimal curves. The notation is mainly vector analysis.

The second chapter, from page 85 to page 313, deals with the first fundamental form of surface theory. We find here an introduction into the algebra and analysis of tensors, which may have its independent use. It is applied to conformal mapping, geodesics, Gaussian curvature and applicability of surfaces.

The third chapter, from page 314 to page 434, discusses the second fundamental tensor. This leads to spherical representation, Dupin indicatrix, lines of curvature, asymptotic lines, and construction of surfaces with given fundamental tensor or tensors.

The last chapter, pages 435–560, contains special surfaces, as ruled, Weingarten, minimal, spherical, pseudospherical, Monge and Joachimsthal surfaces. A carefully composed index closes the book.

Among the many points of interest we may mention the discussion of geodesic mapping, the construction of surfaces from the spherical image of conjugated and asymptotic curves, Bonnet's theorem on the determination of a surface with given fundamental tensors, and a formula of all surfaces with given metrical tensor. A trivalent symmetrical tensor stands godfather to ruled surfaces. The concluding articles deal with surfaces with one congruence and with two congruences of plane lines of curvature, and end with Dupin cyclids. *D. J. Struik* (Cambridge, Mass.).

Cameron, E. A. On loci associated with osculants and penosculants of a plane curve. Amer. Math. Monthly **46**, 341–345 (1939). [MF 159]

The author proves a general theorem concerning the location of a point P associated with an osculating curve of a plane curve C at one of its points M. The point P must lie on the envelope of the locus of certain points Q related in a given manner to P and the penosculating curves of C at M.

Applications of the theorem are then considered, and also a sort of dualism between the elements of the general theorem. *V. G. Grove* (East Lansing, Mich.).

Ionesco, D. V. Quelques applications des propriétés d'une famille de trajectoires. Bull. Soc. Sci. Cluj **9**, 294–298 (1939). [MF 74]

A point mass starts from rest on the upper part of a conic situated in a vertical plane, the focal axis being horizontal. It is required to find the point where the mass leaves the conic. This problem for a parabola is stated in P. Appell: Traité de Mécanique Rationelle, vol. 1, p. 479. The author finds that for conics with the same focus and directrix and masses with the same initial height, the height of leaving is independent of the eccentricity. This property enables the author to construct the center of curvature of any conic, utilizing in the process a classical construction for the center of curvature of a parabola. *E. W. Titt.*

Ionesco, D. V. Généralisation d'une propriété mécanique de la chaînette. Bull. Soc. Sci. Cluj **9**, 299–306 (1939). [MF 75]

The center of gravity of an arc AB of a catenary has the same abscissa as the point of intersection of the tangents at A and B [see reference to P. Appell in the above review, p. 170]. The author finds that the catenary is the only curve having this property for an arbitrary arc. The paper then considers generalizations of the catenary, that is, curves with a variable density $\rho(x, y, y')$ having the above property for an arbitrary arc. *E. W. Titt* (Hyattsville, Md.).

Simonart, Fernand. Enveloppes et congruences associées à une courbe. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 130–138 (1939). [MF 107]

The general problem treated in this paper is as follows: Given a curve Γ defined intrinsically by $R(s)$ and $T(s)$ $(R, T\neq 0)$ where s is the arc; with the moving trihedral of Γ is associated a one parameter family of surfaces $F(X, Y, Z, \omega)=0$: to investigate the characteristics, envelopes, edges of regression and congruences generated by F. For a fixed ω the general equations are

$$F=0, \quad \phi\equiv\sum F_x\frac{\delta x}{\delta s}=0, \quad \sum\phi_x\frac{\delta x}{\delta s}+\phi_\omega=0,$$

$\delta x/\delta s$ being given by Cesàro's equations with $dx/ds=0$. Similar equations hold if ω is a function of the arc. These equations are then applied to the special cases of planes through either the tangent, normal or binormal determined by an angle ω. *M. S. Knebelman* (Pullman, Wash.).

Pailloux, Henri. Sur l'enveloppe des surfaces inextensibles. C. R. Acad. Sci. Paris **209**, 338–340 (1939). [MF 235]

In this note, the author reviews known properties and gives new ones of certain curves related to the envelope of a rigid surface. He also extends the work of M. R. Garnier [C. R. Acad. Sci. Paris **208**, 1553 (1939)] to the envelopes of simply inextensible surfaces. The attack is made from the kinematic point of view. *V. G. Grove.*

***Finikoff, M. S. Déformation à réseau conjugué persistant et problèmes géométriques qui s'y rattachent.** Mémorial des Sciences Mathématiques, v. **96**. Gauthier-Villars, Paris, 1939. 86 pp.

In six short chapters, the author gives a review of the work of himself and numerous other mathematicians on the subject of deforming or otherwise transforming a surface in euclidean space of three dimensions, such that a conjugate net transforms into a conjugate net.

The first chapter concerns itself with the analytic formulation of the problem, its relation to the spherical indicatrix of the conjugate net, and to geodesics. The chapter concludes with special cases and applications.

The second chapter relates the problem to quadratic solutions of a Laplace or a Moutard equation and gives a discussion of a transformation of Peterson under which conjugacy is preserved. The fundamental transformation F of Eisenhart is given as an example of a transformation preserving a conjugate net.

The third chapter discusses cyclic congruences. After a convenient definition of the deformation of a congruence, a treatment is given of the problem of deforming a congruence so that the developables of the congruence shall deform into developables. The deformation of congruences related in special ways to a surface is discussed; the relation of the transformation of Ribaucour to the problem is shown.

The fourth chapter deals with the relation of surfaces and congruences of Bianchi and the transformation of Bianchi to the persistence of conjugate nets.

The fifth chapter is devoted to the work of Gambier, Voss, Guichard, Eisenhart, Egoroff and others on the deformation of surfaces of Voss-Guichard. The problem of determining two applicable surfaces such that the asymptotic curves of one correspond to a conjugate net on the other is considered.

The final chapter concerns itself with the question of the deformation of certain nets called kinematically conjugate by Bianchi.

There is attached a bibliography of over one hundred fifty references. *V. G. Grove* (East Lansing, Mich.).

Lane, E. P. and MacQueen, M. L. The curves of a conjugate net. Duke Math. J. **5**, 692–704 (1939). [MF 184]

In this paper the authors developed power series expansions in nonhomogeneous projective coordinates for the curves of a conjugate net. Using these expansions the equations of quadrics having second order contact with the sustaining surface of the net were derived. By further specialization there is found to be a pencil of quadrics having third order contact with the curves of the net. There is a unique quadric having fifth order contact with one of the curves of the net. In terms of these quadrics, new geometric characterizations are given for the principal point and principal join as defined by Bompiani.

Analytic conditions that one of the family of curves of a conjugate net be twisted cubics are derived. In the last two sections the curves of the net are specialized to be Darboux and Segre curves, and finally one of the families of curves to be plane curves. *V. G. Grove.*

Cartan, Elie. Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques. Math. Z. **45**, 335–367 (1939). [MF 45]

An investigation of the existence of families of isoparametric hypersurfaces in a spherical space (space of constant positive curvature) of an arbitrary number of dimensions. It is shown that there exist families of isoparametric hypersurfaces having three distinct principal curvatures in spherical spaces of 4, 7, 13 and 25 dimensions and that such families exist only in spherical spaces of these dimensions. The following general theorem is proved: If in a spherical space there exists a family of isoparametric hypersurfaces having p distinct curvatures of the same degree of multiplicity, the general equation of these hypersurfaces is of the form $P(x_1, x_2, \cdots, x_{n+1})=\cos pt$ where P is a harmonic polynomial of degree p satisfying the condition

$$\sum_i\left(\frac{\partial P}{\partial x_i}\right)^2=p^2(x_1^2+x_2^2+\cdots+x_{n+1}^2)^{p-1}.$$

Finally it is pointed out that the investigation of families of isoparametric hypersurfaces for which all the principal curvatures have the same degree of multiplicity is reducible to an algebraic problem. *T. Y. Thomas.*

Allendoerfer, Carl B. Rigidity for spaces of class greater than one. Amer. J. Math. **61**, 633–644 (1939). [MF 20]

Let U_n be a simply connected domain of a Riemann space of dimensionality n such that U_n can be imbedded in the Euclidean space E_{n+p}, the imbedding functions $y^i(x)$ being of class C^4. Under the assumption that the first normal space is of the same dimension q(not greater than p) at

each point of U_n, an integer, called the type of U_n at P, is defined at every point P of U_n. In the case of a hypersurface for which $q=1$ the type is the same as the rank of the second fundamental form. The following result is proved: If U_n is of type not less than 3 at every point it can be imbedded in an E_{n+q} and this imbedding is unique to within a rigid motion. Also, if the type of $U_n \geqq 4$ at every point, the Codazzi equations are consequences of the equations of Gauss [cf. T. Y. Thomas, Acta math. **67**, 169–211 (1936), where this result is proved for spaces of class one]. In addition several related theorems and lemmas are proved among which is a theorem giving sufficient algebraic conditions for the imbedding of U_n in the Euclidean space E_{n+q}.

T. Y. Thomas (Los Angeles, Calif.).

Craig, H. V. On extensors and a Euclidean basis for higher order spaces. Amer. J. Math. **61**, 791–808 (1939). [MF 41]

In this paper the author gives a Euclidean representation of spaces of higher order and then continues the development of his theory of extensors (tensors of spaces of higher order) as previously discussed [Amer. J. Math. **59** (1937)].

Consider a space X_N of N dimensions with coordinate systems x^a and $\bar{x}^l$, $a, l=1, \cdots, N$. If x^a are given as functions of a parameter t, then a curve Γ of, say, class $(M+1)$ is determined in X_N. Associated with this curve are the quantities defined by

$$(1)\quad \begin{aligned} &x^{1\alpha}=dx^a/dt; \quad x^{\alpha\alpha}=d^\alpha x^a/dt^\alpha, && \alpha, \beta=0, \cdots, M;\\ &X_l{}^a=X_{0l}{}^{0a}=\partial x^a/\partial \bar{x}^l; \quad X_{\lambda l}{}^{\alpha a}=\partial x^{\alpha a}/\partial \bar{x}^{\lambda l}, && \alpha \geqq \lambda. \end{aligned}$$

The set of $N(M+1)$ numbers $x^{\alpha a}$ are said to define an extended point (expoint). Furthermore, the totality of such points determine a space of higher order $X_{N(M+1)}$.

If y^i, $i=1, \cdots, L$, $L>N$ are the coordinates of a Euclidean space E_L, then the equations

$$(2)\qquad y^i=y^i(x^a, x^{1a}, \cdots, x^{Ma})$$

define a representation of $X_{N(M+1)}$ onto E_L. Associated with each curve in X_N is a "warped curve" in E_L. The totality of all warped curves is defined as an exsurface in E_L.

The author then classifies the expressions y^i as follows: (1) some y^i not invariant in functional form under an admissible parameter change; (2) all y^i invariant. The geometry of E_L is now studied for the first class of expression. First the metric tensor is formed

$$(3)\qquad g_{a\alpha,\, b\beta}=\sum_{i=1}^{L}\partial y^i/\partial x^{a\alpha}\cdot \partial y^i/\partial x^{b\beta}, \qquad a, b=1, \cdots, N.$$

It is shown that, for $|g|$ not to vanish, $L \geqq N(M+1)$. The tangent exvector of $X_{N(M+1)}$ is then defined to be $x^{\alpha+1\, a}$, $\alpha=0, \cdots, M$. The square of the magnitude of the tangent exvector is equal to the square of the derivative of arc length in E_L with respect to t. Finally, two types of normals to the exsurface in E_L are defined. The section concludes with a study of some special exsurfaces. For class (2), the author shows that $|g|$ vanishes.

The next section continues the development of extensor theory in $X_{N(M+1)}$. Christoffel symbols are formed with respect to (3). Again the author shows that Christoffel symbols can be formed with respect to a minor of the $|g|$ determinant (of reduced range). Finally, the law of covariant differentiation of a tensor, one of whose indices is of full range and the other of reduced range, is given.

N. Coburn (Austin, Tex.).

Michal, A. D. General differential geometries and related topics. Bull. Amer. Math. Soc. **45**, 529–563 (1939). [MF 51]

If a Hausdorff space can be mapped locally on a linear space, the elements of the linear space can be considered as a set of coordinates for the corresponding elements in the Hausdorff space. Subject to various specializations, this permits the generalization of major portions of differential geometry. The present paper contains an outline of the previous contributions of the author and his associates, A. H. Clifford, V. Elconin, D. H. Hyers, I. E. Highberg, L. S. Kennison, R. S. Martin, E. W. Paxson, T. S. Peterson and A. E. Taylor to this development. A complete bibliography is presented. Two new additions to the theory are given. A more general notion of "differential" is introduced and compared with the Fréchet differential. However, the greater generality does not seem to extend beyond this point. A generalization of Riemannian geometry based on an invariant differential form and containing such notions as "affine connection," "Christoffel symbols" and "curvature tensor" also appears.

F. J. Murray.

TOPOLOGY

George, Erich. Topologie regulärer Kurvenscharen. Deutsche Math. **4**, 462–476 (1939). [MF 133]

The author studies families of "simple curves" (that is, topological images of the circumference, the closed open and half open segment) on a closed surface. He calls a family σ regular ("reguläre Kurvenschar"), if each point of the surface with the possible exception of a finite number of "singular" points (which do not belong to any curve of σ) is contained in exactly one curve of σ, and, moreover, each regular point has a neighborhood in which σ is homeomorphic to the family of segments $x=\text{const.}$, $0 \leqq y \leqq 1$ in the (x, y)-plane. The author shows that, given any regular family σ, there exists at least one "transversal" regular family whose curves have no more than one point of intersection with arcs of σ in a sufficiently small neighborhood of any regular point. Furthermore, the author considers "polygonal regions," that is, regions bounded by a finite number of arcs belonging to a given regular family σ and a given transversal family σ'. He assigns to regions of this type "covering surfaces" ("Überlagerungsflächen"). This rather complicated notion seems to have little to do with the usual concept of the covering surface.

W. Hurewicz (Chapel Hill, N. C.).

Kelley, J. L. Fixed sets under homeomorphisms. Duke Math. J. **5**, 535–537 (1939). [MF 171]

Consider a (compact) continuum M and a homeomorphism T of M into a subset of itself. It is proved, with the help of Brouwer's Reduction Theorem, that there is a subcontinuum H of M which has no cut points (of itself) and is such that $T(H)=H$. It follows that if there is no fixed point under T then there is a 0th order cyclic element of M (that is, a subset of M which is maximal with respect to the property of being a subcontinuum without cut points) which contains its image under T. Another consequence is: If every homeomorphism of every 0th order cyclic element of M has a fixed point, then every homeomorphism of M has a fixed point. If M is locally connected these theorems reduce to known results of Ayres and Borsuk.

R. H. Fox (Princeton, N. J.).

Dieudonné, Jean. Un exemple d'espace normal non susceptible d'une structure uniforme d'espace complet. C. R. Acad. Sci. Paris **209**, 145–147 (1939). [MF 224]

The ordinal numbers of the second class, in the topology induced by their ordering, are completely normal and have a unique uniform structure in which they fail to be complete. *J. W. Tukey* (Princeton, N. J.).

Halperin, Israel. Additivity and continuity of perspectivity. Duke Math. J. **5**, 503–511 (1939). [MF 168]

Let Ω denote a cardinal number and a_α and b_α, for $\alpha<\Omega$, elements of an irreducible continuous geometry. If

$$\sum_{\beta<\Omega}\prod_{\beta\leq\alpha<\Omega}a_\alpha=\prod_{\beta<\Omega}\sum_{\beta\leq\alpha<\Omega}a_\alpha=a,$$

define $\lim_{\alpha\to\Omega}a_\alpha$ as a. If $a_\alpha\sim b_\alpha$ for all $\alpha<\Omega$ and if $\lim_{\alpha\to\Omega}a_\alpha=a$ and $\lim_{\alpha\to\Omega}b_\alpha=b$, then $a\sim b$. If $a_\alpha\sim b_\alpha$ and the a_α's are independent and the b_α's are also independent, then

$$\sum_{\alpha<\Omega}a_\alpha\sim\sum_{\alpha<\Omega}b_\alpha.$$

These results are new for $\Omega>\aleph_0$ and the proofs are more direct. *F. J. Murray* (New York, N. Y.).

Smith, P. A. Transformations of finite period. II. Ann. of Math. **40**, 690–711 (1939). [MF 105]

This paper extends in three directions previous results of the author [Ann. of Math. **39**, 127–164 (1938); later referred to as I]. Define M as a bicompact Hausdorff space such that every open set in M is an F_σ, and the Menger-Urysohn dimension of M is finite; L is the set on M left invariant by a topological transformation T of M into itself having finite period p; $\mathfrak{f}$ is a field to serve as coefficient field for Čech cycles and homologies.

When M is specialized to be a 3-sphere S_3, I shows that, for p a prime, L is either empty, two points, a simple closed curve or a set of dimension 2 with the mod 2 Betti numbers of S_2. It is here shown, first for p a prime, that L, if of dimension 2, is an S_2 by proving that L is a generalized closed manifold in the sense of Wilder [Ann. of Math. **35**, 876–903 (1934)]. There Wilder has shown that a g.c.m. in S_3 is a closed surface, so L is an orientable closed surface and, because of its Betti numbers, an S_2. For arbitrary period $p=ts$, t prime, if μ is the invariant set of T^s, the problem is reduced to finding the invariant sets of μ under periodic homeomorphisms in case μ is (i) a circle, (ii) an S_2. For (i) see Ritt [Ann. of Math. **17**, 113–122 (1915)]; for (ii) see Brouwer [Math. Ann. **80**, 39–41 (1919)] or Kerekjártó [Math. Ann. **80**, 36–38 (1919)]. Smith's result is: When T is of prime or composite period p and preserves (reverses) orientation, L is either null or a simple closed curve (L is either an S_2 and $p=2$, or L consists of two points and p is even).

A further extension of I is gotten by placing only local conditions on the general M. Property P relative to $\mathfrak{f}$ at the point d of M is the existence, for every neighborhood $D(d)$ and some m, of an m-cycle Ω mod $M-D$ such that for no open subset $D'\neq 0$ of D is $\Omega\sim 0$ mod $M-D'$; and for every neighborhood $B(b)\subset D$ there is a neighborhood $B'(b)\subset B$ such that Ω is a basis for m-cycles mod $M-B$ relative to homologies mod $M-B'$; while every cycle mod $M-B$ of dimension not equal to m is equivalent to 0 mod $M-B'$. If M has property P at each point d, M is of type P. If, further, the number m is the same for each d, M is of type P_m. When M is of type P_m the local Betti numbers of Čech at d (and if M is metric the Betti numbers in d of Alexandroff) are the same as those at the points of an m-cell. Property Q at d relative to $\mathfrak{f}$ is the existence for every $D(d)$ of a $D'(d)$ such that for every $B(b)\subset D'$ there is a $B'(b)\subset B$ for which γ a cycle mod $M-A$ in $M-B$ implies $\gamma\sim 0$ mod $M-A'$ in $M-B'$. If M has Q at every d, it is of type Q. The results are:

1. If M is connected and of types P_m and Q, p is prime, $\mathfrak{f}=\mathfrak{p}$ (the field of integers mod p), then L is of types P and Q.

2. If p is either prime or composite, and M is of types P_m and Q relative to $\mathfrak{t}$, t a prime factor of p, then L is nowhere dense in M.

3. A. If $p=t^n$, t prime, and M is both connected and HLC^n relative to $\mathfrak{t}$ ($n=\dim M$), then L has a finite number of components, each HLC^n.

3. B. If, in addition, M is of types P_m and Q (relative to $\mathfrak{t}$), each component L_i of L is of types P_{r_i} and Q.

3. C. If it be further given that M is metric, the 0 and 1-dimensional components of L are points and simple closed curves.

3. D. If, in addition to the hypotheses of 3. B., M is restricted to be an m-dimensional quasi-manifold, $n=m$, $\alpha=1$, then the Betti numbers $B_{r_i}(L_i)\geqq 1$. These results are obtained by extension of the method of I in which an important role is played by ρ-cycles and $\leqq$ homologies.

Finally Smith shows that, if T is regular and has odd period and M is compact, locally euclidean and orientable, then each component of L is orientable. A regular T is one for which L is locally euclidean. Hence $B_{r_i}(L_i)=1, 0$ according as L_i is orientable or not, $r_i=\dim L_i$ [Flexner, Ann. of Math. **32**, 393–406, 539–548 (1931)]; by 3. D. above, $B_{r_i}(L_i)=1$ for prime period. An induction on the dimension of M removes the restriction on the period.

W. W. Flexner (Princeton, N. J.).

Whyburn, G. T. The existence of certain transformations. Duke Math. J. **5**, 647–655 (1939). [MF 180]

A transformation f of a continuum A into a continuum B is said to be non-alternating if for any two points b_1 and b_2 of B the set $f^{-1}(b_1)$ does not separate any two points of $f^{-1}(b_2)$ in A; it is said to be interior if $f(U)$ is an open subset of B for every open U in A. The chief result is to show that, in order that a non-alternating interior transformation f exist carrying a locally connected continuum A into a linear interval B, it is necessary and sufficient that A be a cyclic chain, that is, A contain two points x and z such that every point y of A lies on an arc of A with end points x and z. In proving the existence of the desired transformation f, it is shown that f can be chosen as monotone ($f^{-1}(b)$ connected for every $b\varepsilon B$) in addition if A is unicoherent or f can be light ($f^{-1}(b)$ totally disconnected) if A is 1-dimensional. Finally any connected graph can be mapped on a linear interval by an interior transformation and a tree A can be so mapped if and only if A has no infinite branch points and the set of end points of A is closed except for a finite set. *W. L. Ayres* (Ann Arbor, Mich.).

Wallace, A. D. Some characterizations of interior transformations. Amer. J. Math. **61**, 757–763 (1939). [MF 35]

It is shown that if S and S_1 are metric spaces and $T(S)=S_1$ is continuous, the following conditions are each equivalent to the property of interiority for T: (i) for any sequence of sets Y_n converging to Y in S_1, $\lim T^{-1}(Y_n)=T^{-1}(Y)$; (ii) for any set Y in S_1, $T^{-1}(\bar{Y})=\overline{T^{-1}(Y)}$; (iii) for any set X in S, $T(X^0)=T(X)^0$, where X^0 denotes the set of all interior points of X. Also a study is made of transformations, called

strongly continuous, satisfying $T(X)=\lim T(X_n)$ for every sequence of sets X_n converging to X in S and of transformations, called normal, such that there exist arbitrarily small neighborhoods of any point whose boundaries transform into boundary points of the images of the neighborhoods.
G. T. Whyburn (Charlottesville, Va.).

Puckett, W. T., Jr. Concerning local connectedness under the inverse of certain continuous transformations. Amer. J. Math. **61**, 750–756 (1939). [MF 34]

It is shown that if A and B are compact and $T(A)=B$ is an interior transformation, the property of local connectedness will be invariant under T^{-1} if and only if for every $\epsilon>0$, a $\delta>0$ exists so that any two points a, $a'\epsilon A$ with $T(a)=T(a')$ and $\rho(a,a')<\delta$ lie together in a connected subset of A of diameter less than ϵ. Applications are made to the cases where T is 0-regular and where T is semi-monotone (that is, the inverse of each continuum in B has a finite number of components) and to the related cases where the inverse of each point in B is finite or countable.
G. T. Whyburn (Charlottesville, Va.).

Kelley, J. L. A metric connected with property S. Amer. J. Math. **61**, 764–768 (1939). [MF 36]

In answer to a question proposed by the reviewer, the author shows that if M is a metric connected space having property S (that is, it is the sum of a finite number of arbitrarily small connected sets), then M is homeomorphic with a space M^* having property S, and in which every spherical neighborhood of any point is connected and has property S.
G. T. Whyburn (Charlottesville, Va.).

Whyburn, G. T. Semi-locally connected sets. Amer. J. Math. **61**, 733–749 (1939). [MF 33]

A connected metric space M is defined to be semi-locally-connected (s.l.c.) if for each $p\epsilon M$ and $\epsilon>0$ there exists $\delta>0$ such that $M-S(p,\epsilon)$ is contained in finitely many components of $M-S(p,\delta)$. The compact Peano spaces are special cases of s.l.c. spaces and the results obtained generalize in many cases classic theorems on Peano spaces. For instance, the structure of an s.l.c. space, studied from the standpoint of its cyclic element decomposition, is found to be closely analogous to that of a Peano space; in particular, every compact s.l.c. space M has a topological image in a Peano space P such that the intersection of M with a cyclic element of P is a cyclic element of M, and a number of properties are cyclicly extensible and reducible. The "arc theorem" of Peano spaces does not extend to s.l.c. spaces. However, if M is a compact s.l.c. space M in the plane, then the boundary of each complementary domain of M is Peanian. *R. L. Wilder* (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

Hitchcock, Frank L. Algebraic equations with complex coefficients. J. Math. Phys. Mass. Inst. Tech. **18**, 202–210 (1939). [MF 192]

If

$$x^3+(a_2+b_2i)x^2+(a_1+b_1i)x+(a_0+b_0i)=0$$

has a pure imaginary root iy_1, $x-iy_1$ is a factor, and y_1 satisfies

$$(1)\qquad y^3+b_2y^2-a_1y-b_0=0,\quad a_2y^2+b_1y-a_0=0.$$

The greatest common divisor of (1) gives y_1. Similarly for degrees greater than 3.

It is not noted that for μ imaginary roots $iy_1,\cdots,iy_\mu$ the greatest common divisor gives the polynomial of degree μ with zeros $y_1,\cdots,y_\mu$. This must be considered in evaluating the usefulness of the method. A root $\alpha+i\beta$, $\alpha\neq0$, will cause a remainder when the divisor process is applied to (1). A method of trial is used to change the given equation to one with a root $\epsilon+i\beta$, ϵ close to zero. Then the process above for pure imaginary roots is applied. The method is applied in detail to a cubic equation with numerical complex coefficients. Its application to equations of degree greater than 3 is outlined.

It is doubtful to the reviewer whether the method represents saving of labor in practice. *A. J. Kempner.*

Van den Akker, J. A. A mechanical integrator for evaluating the integral of the product of two functions and its application to the computation of I.C.I. color specifications from spectrophotometric curves. J. Opt. Soc. Amer. **29**, 364–369 and **Erratum**, 501 (1939). [MF 187]

The instrument consists of an input board, a multiplier, and an integrator. The input board is advanced as x increases. Two follower rods R, R', constrained to move at right angles to the X-axis, are made to follow the graphs of the given functions y, y' by two operators. The multiplier consists of a beam, rotatable about a fixed axis O, forming with the X-axis two similar right triangles AOB and aOb. Rod R is connected to leg AB and R' to hypotenuse aO in such a way that $AB=y$ and $aO=y'$; OB has fixed length, say 1; hence $ab=yy'$. The integrator is a conventional variable speed drive consisting of a horizontal table T, which is made to rotate through an angle dx, and a counting wheel in contact with T at a distance $ab=yy'$ from the center of rotation of T. Mechanical features are the transmission of linear displacements by means of thin steel tapes operating against springs, and the use of a spot of light rather than a pointer to follow the curves. An accuracy of better than 0.2 percent is obtained. *P. W. Ketchum.*

Beevers, C. A. A machine for the rapid summation of Fourier series. Proc. Phys. Soc. **51**, 660–663, discussion 664–667 (1939). [MF 206]

One method for the summation of Fourier series is the use of the printed strips advocated by H. Lipson and the author [Proc. Phys. Soc. **48**, 772–780 (1936)]. These strips carry the values at given intervals of a set of sine curves; the process consists in the selection of the appropriate strips and the addition of the columns of numbers thus obtained. It is proposed to construct a machine to work on this same principle. First, the fundamental sine curve of any given amplitude is represented by a number of sets of electrical impulses. A switch can then arrange these impulses to correspond to a curve of any desired wave-number of this same amplitude. The sets of impulses are then fed into electro-magnetic counters. The discussion concerns the accuracy obtained by this method, as well as the relative merits of the proposed machine and the standard punched-card machines. *R. M. Foster* (New York, N. Y.).

Meissner, W. Tafel der ln Γ-Funktion mit komplexem Argumentbereich. Deutsche Math. **4**, 537–555 (1939). [MF 135]

The paper contains three tables of $\ln\Gamma(z)$. Table 1 gives

fifteen place values at the 24 points $z=(a+b\sqrt{3}i)/8$; $a=4, 6, 8, 10$; $b=0, 2, 4$; $a=5, 7, 9, 11$; $b=1, 3, 5$. These points form a triangular network in the rectangle $\frac{1}{2}\leqq\Re(z)<\frac{3}{4}$, $0\leqq\Im(z)\leqq\frac{3}{4}\sqrt{3}$. Table 2 gives seven place values at the 192 points obtained by adding the center of each triangle in the network and also the center of each smaller triangle thus formed. Table 3 gives the polar values at the same 192 points. The author shows that linear interpolation in these tables is inaccurate, but that the Lagrange interpolation formula yields good results. As an example of such interpolation, he computes values for the logarithmic derivative of $\Gamma(z)$ at a number of intermediate points. Values of $\ln\Gamma(z)$ for any point outside the range of these tables may be found by use of the difference equation for $\Gamma(z+n)$, together with the multiplication formula, which expresses $\Gamma(z)$ in terms of $\Gamma(z+s/n)$, $s=0, 1, \cdots, n-1$. The author's statement that there were no previous tables of the Γ-function for complex arguments might be questioned in view of H. T. Davis' table of $1/\Gamma(re^{i\theta})$ for various values of r and θ [Tables of the higher mathematical functions, Bloomington, Ind., pp. 269–274 (1933)]. *P. W. Ketchum.*

Lowan, A. N. On the computation of the second differences of the Si(x), Ei(x), and Ci(x) functions. Bull. Amer. Math. Soc. **45**, 583–588 (1939). [MF 55]

If $f(x)$ denotes any of the above functions, it is shown that, when $x>0.1$, the second difference at the tabular interval $h=10^{-4}$ agrees to 12 decimal places with $\frac{1}{2}h[f'(x+h)-f'(x-h)]$. The computation of second differences by linear interpolation between values of the second differences computed at an interval greater than 10^{-4} is also considered. *L. M. Milne-Thomson.*

Witt, G. Über die Berechnung der Perioden der elliptischen Integrale. Astr. Nachr. **269**, 68–73 (1939). [MF 137]

In the Weierstrass theory of elliptic integrals there is no simple method of calculating the periods, in terms of the numbers g_2, g_3, of the normal forms

$$\int(4s^3-g_2s-g_3)^{-\frac{1}{2}}ds, \quad \int s(4s^3-g_2s-g_3)^{-\frac{1}{2}}ds.$$

In 1875 H. Bruns [Jubelschrift. Dorpat] obtained an expression for the periods in terms of certain hypergeometric series of the form $F(\alpha, \beta, 1, 1-1/g)$, where $g=g_2^3/27g_3^2$; but these series converge too slowly for practical purposes unless g is very near 1. In an earlier paper [Astr. Nachr. **261**, 429 (1937)], Witt obtained for $g>1$ more rapidly converging series by means of appropriate substitutions on g. In the present paper he obtains series with satisfactory convergence properties for all values of $-\infty<g<\infty$. Values of g are divided into five ranges and the periods in each case expressed in terms of a total of ten different hypergeometric functions. The logarithms of the coefficients of the first 6 to 8 terms of the author's rapidly converging series for each of these ten functions are tabulated. *P. W. Ketchum* (Urbana, Ill.).

Gyllenberg, W. Über eine graphische Lösung einer Integralgleichung. Astr. Nachr. **269**, 52–53 (1939). [MF 8]

The author applies a previously described method [Meddelanden Lund Astron. Observ. Ser. I, Nr. **145**] for solving integral equations of the form

$$\phi_1(x)=\int_{-\infty}^{\infty}\phi_2(x-y)\phi_3(y)dy$$

to the problem of the distribution of star sizes. Results differ from those obtained by Strömberg [Astrophys. J. **71**, 163 (1930)] by a numerical method. The author states that the unknown function ϕ_3 does not have an oscillatory character as found by Strömberg, and that one cannot conclude as a consequence that star matter is divided among several components of giant and dwarf stars. The discrepancy between the two methods is assigned to the fact that if ϕ_1 vanishes except for one value of x, the corresponding ϕ_3 is oscillatory. A numerical method like Strömberg's, which reduces the problem for a continuous ϕ_1 to a sum of a finite number of these elementary ϕ_3's, will usually give an oscillatory solution, while the true solution may be non-oscillatory. *P. W. Ketchum* (Urbana, Ill.).

Joseph, J. A. On the coefficients of the expansion of $X^{(n)}$. Ann. Math. Statistics **10**, 293–296 (1939). [MF 154]

A linear connection between the Bernoulli numbers seems to be the only result not either well known or obvious from the definitions, and this was given by Blissard in 1861 [Quart. J. Math. **4**, 284]. *C. C. Craig.*

Baeschlin, C. F. Die Fehlertheorie der Herablegung eines Hochpunktes. Schweiz. Z. Vermessgswes. **37**, 199–206 (1939). [MF 185]

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Portugaliae Mathematica 1, parte 2, fasc. 1. Lisboa, 1939.

The second part of each volume of Portugaliae Mathematica will contain reprints of the mathematical papers of Portuguese mathematicians. This number contains 11 papers of Aureliano de Mira Fernandes published in the Rendiconti della R. Accademia dei Lincei over the period 1928–1937.

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